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APPLICATION OF AN EXACT METHOD FOR THE SOLUTION OF SYMMETRICAL FIXED END CIRCULAR ARCHES

ALLEN W. ABBOTT

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APPLICATION OF AN EXACT METHOD FOR THE SOLUTION OF SYMMETRICAL FIXED END CIRCULAR ARCHES

BY

Allen W. Abbott

Submitted to the Faculty of Rensselaer
Polytechnic Institute in partial fulfillment
of the requirements for the degree,
Master of Civil Engineering

Troy, New York
May, 1950

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A14

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DEFINITION OF SYMBOLS

$\frac{M}{EI}$ diagram -- the moment diagram of a particular reaction or load, divided by the modulus of elasticity, E, and also by the cross sectional moment of inertia, I.

L -- length of span in feet.

h -- rise in feet, from springing line to crown.

w -- distance in feet from crown to load point, measured parallel to springing line.

H -- horizontal thrust in kips.

M_R -- fixed end moment in foot kips at right springing.

M_L -- fixed end moment in foot kips at left springing.

V_R -- vertical reaction at right springing.

V_L -- vertical reaction at left springing.

x, y, z -- coordinate variable distances, in feet.

$x^2 + y^2 = r^2$ -- equation of circular arch.

$$r = \frac{4h^2 + L^2}{8h}$$

A -- area of $\frac{M}{EI}$ diagram of designating subscript.

$M\text{-}_x$ -- first moment of $\frac{M}{EI}$ diagram of designating subscript about X reference axis which is parallel to springing line.

$M\text{-}_y$ -- first moment of $\frac{M}{EI}$ diagram of designating subscript about Y reference axis which is center line of arch.

Designating Subscripts:

p -- referring to composite $\frac{M}{EI}$ diagram of unit load and vertical reactions.

M_R -- referring to $\frac{M}{EI}$ diagram of fixed end moment at right springing.

M_L -- referring to $\frac{M}{EI}$ diagram of fixed end moment at left springing.

H -- referring to $\frac{M}{EI}$ diagram of horizontal thrust.

f -- distance in feet along Y axis from X axis to centroid of $\frac{M}{EI}$ diagram of unit load and vertical reaction or $\bar{y}_p$.

d -- distance in feet along X axis from Y axis to centroid of $\frac{M}{EI}$ diagram of unit load and vertical reaction or $\bar{x}_p$.

c -- $\bar{x}_{ML}$ or $\bar{x}_{MR}$

m -- $\bar{y}_{ML}$ or $\bar{y}_{MR}$

n -- $\bar{y}_H$

$K = \frac{h}{L}$ -- ratio of rise to span.

$v = \frac{w}{L}$ -- ratio of distance of load point from crown to span.

C_1 to C_{19} -- constants

I_x -- cross sectional moment of inertia at any point along the arch.

I_c -- cross sectional moment of inertia at crown.

α -- angle between tangent to working line of arch, and a line parallel to the springing line.

ds -- increment of arch length.

dx -- increment of length along springing.

K_1 to K_{13} -- constants.

a -- distance in feet from left springing to load point.

b -- distance in feet from right springing to load point.

OBJECT

The object of this thesis is to apply an exact method of indeterminate structures to the solution of influence lines for symmetrical fixed end circular arches, and to study the variation of these influence lines for arches having different ratios of rise to span.

It is furthermore the object of this thesis to compare the results derived herein with similar results previously accomplished for symmetrical fixed end parabolic arches, in an effort to determine if the more simple formulae derived may be practically applied to all parabolic and circular arches with reasonable accuracy.

Two general types of symmetrical fixed end arches will be considered: (1) those with a constant cross sectional moment of inertia, and (2) those with a cross sectional moment of inertia that varies in accordance with some specific function.

INTRODUCTION

This thesis is actually an extension or continuation of a thesis presented to the faculty of Rensselaer Polytechnic Institute in August 1948 by Donald G. Iselin and Albert M. LaLande, entitled, "A Development of an Exact Method for the Solution of Symmetrical Fixed End Parabolic Arches". In their thesis Iselin and LaLande develop the exact solution of symmetrical parabolic arches, with both constant and variable moment of inertia, by application of the principle of the conjugate structure. In the case of arches with a variable moment of inertia, three (3) extremely simple formulae are developed, in terms of the load point, rise, and span of the arch, which may be solved directly for the value of the horizontal thrust and springing moments. Iselin and LaLande further demonstrate for either type of arch that, with a constant ratio of rise to span, the horizontal thrust remains constant and the springing moments vary directly as the span length. Finally, it is shown that the three (3) simple formulae developed for the arch with variable moment of inertia are equally applicable, within practical design limits, to the parabolic arch of constant moment of inertia up to a ratio of rise to span of 0.2.

In this thesis, the principles used by Iselin and LaLande will be applied to the exact solution of symmetrical circular arches of both constant and variable moment of inertia to de-

termine if resulting formulae and relationships between arches of constant rise to span ratio are similar to the results obtained for parabolic arches. The essence of this work is to compare the results obtained for both circular and parabolic arches in order to select a simple set of formulae, or if this is not possible, to draw a simple group of influence line curves which will afford the designer an immediate solution for any symmetrical parabolic or circular arch of either constant or variable cross sectional moment of inertia. Such a solution to be accurate within 5% of the exact solution for any given case.

The principle of the conjugate structure was developed by Dr. J. Sterling Kinney and is fully derived and explained in his text entitled, "Indeterminate Structures". Dr. Kinney is Head of the Structural Division of the Department of Civil Engineering at Rensselaer Polytechnic Institute. The conjugate structure method is actually an extension of the conjugate beam principle applied to single span rigid frames and arches. The following important principles may be stated regarding the conjugate structure:

(1) The conjugate structure, for a given real structure, is identical to the real structure with regard to length of members and their relative positions.

(2) The conjugate structure is positioned in the horizontal plane.

(3) The load which acts in the vertical plane on the conjugate structure, is the $\frac{M}{EI}$ diagram of the real structure.

(4) The shear at any section of the conjugate structure is the slope of the corresponding section of the real structure.

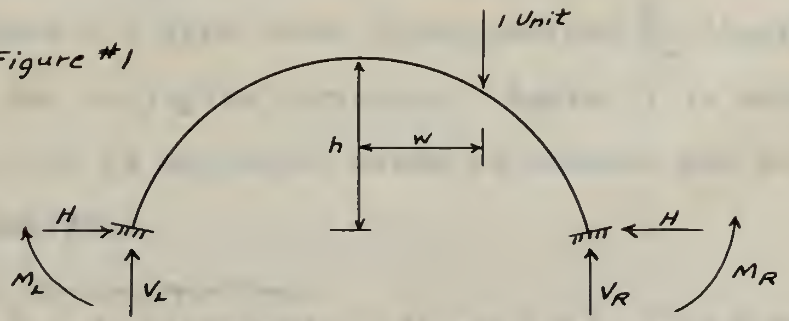
(5) The internal moment on any section of the conjugate structure is the deflection of the corresponding section of the real structure in a direction perpendicular to the lever arm used to find the particular moment.

Accepting the above statements as fact, to solve for the redundants of a fixed end arch, it is necessary to find the area and centroid location of the $\frac{M}{EI}$ diagram for the real structure and to apply certain condition equations which are determined by the given conditions of the problem.

PART I
CONSTANT MOMENT OF INERTIA

In order to develop the exact solution of a symmetrical circular arch it will first be necessary to set up the conjugate structure for the arch loaded with the $\frac{M}{EI}$ diagram of the real structure. The real structure with a unit load acting at any distance w from the crown and the six (6) reactions are shown below in Figure # 1. For the purpose of establishing a sign convention, it will be assumed that forces which cause tension on the inside of the structure will result in positive moment diagrams which will act downward on the conjugate structure. A negative answer for any reaction will indicate that its original assumed direction was incorrect. Since the term EI is constant for the circular arch now under consideration, and it appears in each term of all equations and will therefore eventually cancel out; it will be dropped from all calculations.

Figure #1



The composite moment diagram for the real structure is contained under a curve whose equation is very complex, and therefore its area and centroid location are not readily found by the use of the calculus. Accordingly, the moment diagram

for the real structure is broken into its component simple parts, and the principle of superposition is utilized to determine the composite result. As shown in Figure # 2, the moment diagram is broken into the following four (4) component parts: (1) moment diagram due to the unit load and vertical reactions, (2) moment diagram due to the left springing moment, (3) moment diagram due to the right springing moment, and (4) moment diagram due to the horizontal thrust. Figure # 2 also shows these various $\frac{M}{EI}$ diagrams acting upon the conjugate structure. Again it is mentioned that the EI term is neglected since it cancels out in this particular analysis.

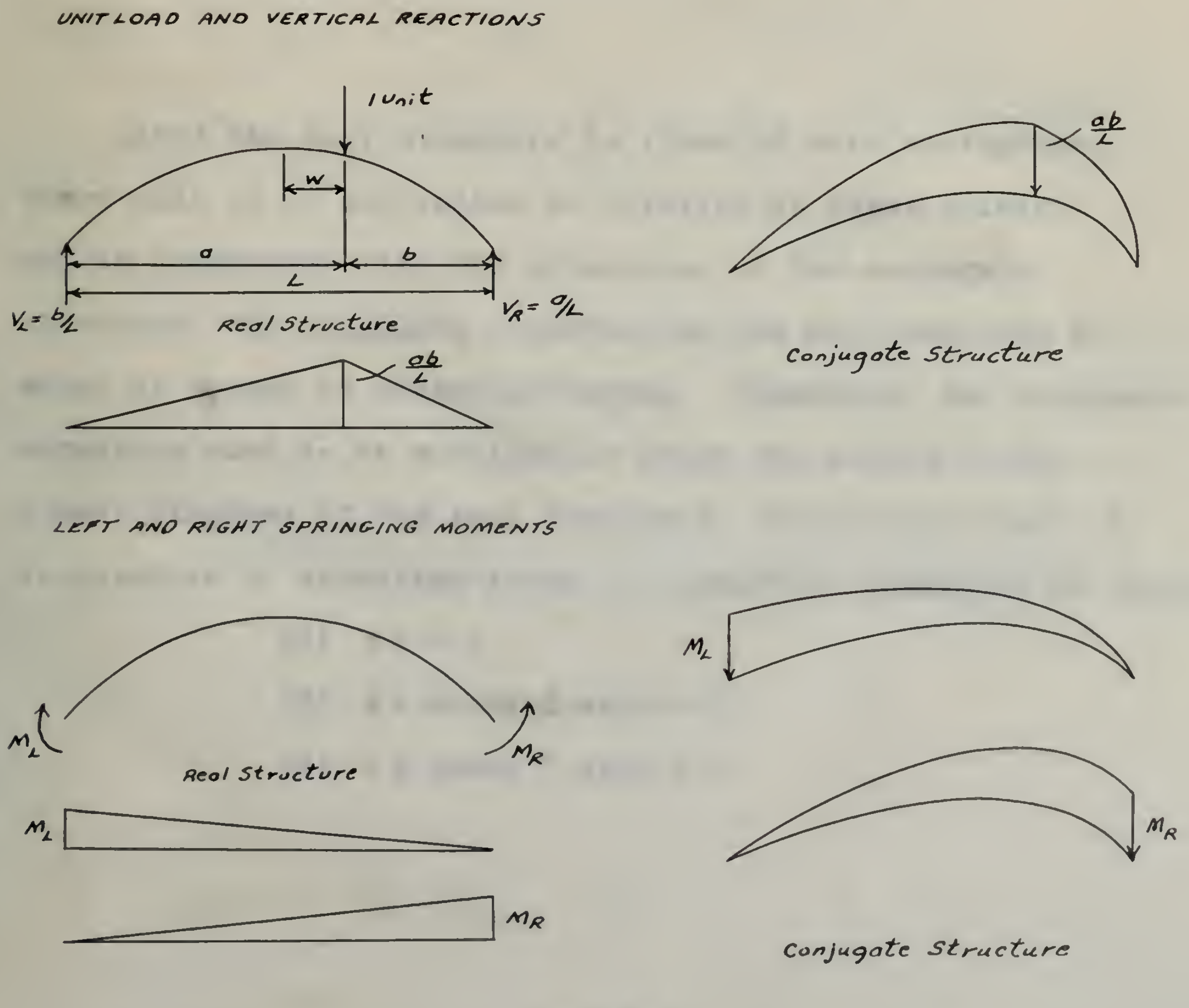


Figure #2

HORIZONTAL THRUST

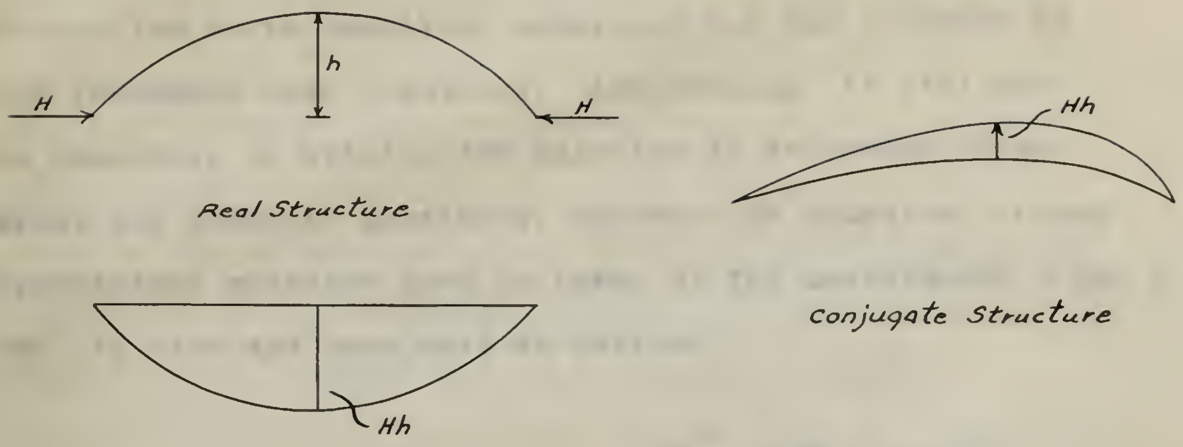


Figure #2

Since the real structure is fixed at both springings, there will be no deflection or rotation at these points; and in accordance with the principles of the conjugate structure, the conjugate structure of the arch can have no shear or moment at either springing. Therefore, the conjugate structure must be in equilibrium under the action of the moment diagrams of the real structure. From these facts it is possible to establish three (3) condition equations as follows:

(1) $\Sigma V = 0$

(2) ΣM about X axis = 0

(3) ΣM about Y axis = 0

If the areas and centroid locations of the various $\frac{M}{EI}$ diagrams of the real structure are known, it is possible to use the above condition equations for the solution of the redundant arch reactions. Accordingly, it will now be necessary to utilize the calculus to determine these areas and centroid locations; however, an equation for any symmetrical circular arch in terms of the coordinates X and Y and its rise and span must be derived.

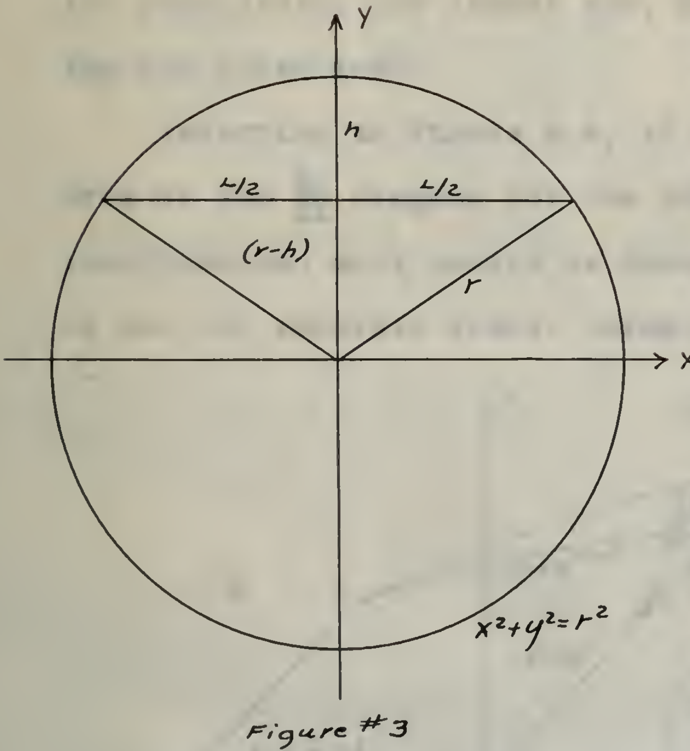


Figure #3

$$r^2 = \left(\frac{L}{2}\right)^2 + (r-h)^2$$

$$r^2 = \frac{L^2}{4} + r^2 - 2rh + h^2$$

$$2rh = \frac{L^2}{4} + h^2$$

$$r = \frac{L^2}{8h} + \frac{h^2}{2h} = \frac{L^2 + 4h^2}{8h}$$

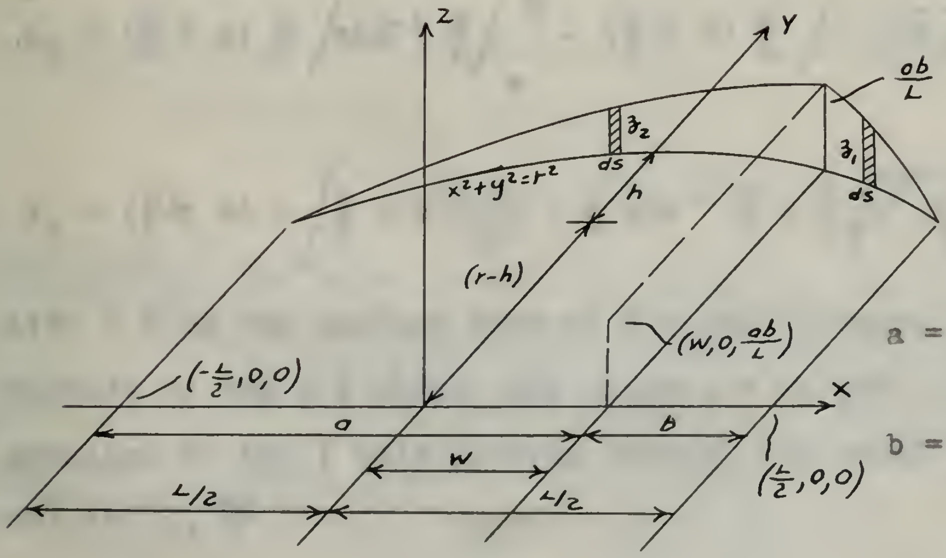
$$r = \frac{L^2 + 4h^2}{8h}$$

Therefore the general equation of a symmetrical circular arch with any rise to span is:

$$x^2 + y^2 = \left(\frac{L^2 + 4h^2}{8h}\right)^2$$

FINDING THE AREAS OF THE $\frac{M}{EI}$ DIAGRAMS will necessitate the use of solid analytical geometry and the calculus. A set of three coordinate axes will be chosen and a right circular cylinder will be symmetrically placed on these axes. In each case, the area of the $\frac{M}{EI}$ diagram desired will be the surface area of a right circular cylinder included in the intersection of three straight planes. The term "r", for radius, will be used in derivation of formulae, since its equivalent in terms of "h" and "L" is too complicated for direct use, and furthermore "r" is constant for any given arch.

Referring to Figure # 4, it will be noted that the area of the $\frac{M}{EI}$ diagram for the unit load and vertical reactions can most easily be determined by finding the area of two (2) separate areas. Area # 1 is the surface area of



$$a = \frac{L}{2} + w$$

$$b = \frac{L}{2} - w$$

Figure # 4

a right circular cylinder bounded by the X-Y plane, the plane $x = w$, and a plane parallel to the Y axis passing through the points $(\frac{L}{2}, 0, 0)$ and $(w, 0, \frac{ab}{L})$.

The equation of plane # 1 is:

$$\frac{x}{a} + \frac{z}{c} = 1$$

$$z_1 = a\left(\frac{1}{2} - \frac{x}{L}\right) \qquad z_1 = \left(\frac{L}{2} + w\right)\left(\frac{1}{2} - \frac{x}{L}\right)$$

$$ds = \frac{r\,dx}{\sqrt{r^2-x^2}}$$

$$dA_1 = \int z_1\,ds$$

$$A_1 = \int_w^{\frac{L}{2}} \left(\frac{L}{2} + w\right)\left(\frac{1}{2} - \frac{x}{L}\right) \frac{r\,dx}{\sqrt{r^2-x^2}}$$

$$A_1 = \left(\frac{L}{2} + w\right) \frac{r}{2} \int_w^{\frac{L}{2}} \frac{dx}{\sqrt{r^2-x^2}} - \left(\frac{L}{2} + w\right) \frac{r}{L} \int_w^{\frac{L}{2}} \frac{x\,dx}{\sqrt{r^2-x^2}}$$

$$A_1 = \left(\frac{L}{2} + w\right) \frac{r}{2} \Big/ \sin^{-1} \frac{x}{r} \Big/_w^{\frac{L}{2}} - \left(\frac{L}{2} + w\right) \frac{r}{L} \Big/ -\sqrt{r^2-x^2} \Big/_w^{\frac{L}{2}}$$

$$A_1 = \left(\frac{L}{2} + w\right) r \left[\frac{1}{2} \sin^{-1} \frac{L}{2r} - \frac{1}{2} \sin^{-1} \frac{w}{r} + \frac{1}{L} \sqrt{r^2 - \frac{L^2}{4}} - \frac{1}{L} \sqrt{r^2-w^2}\right]$$

Area # 2 is the surface area of a right circular cylinder bounded by the X-Y plane, the plane $x = w$, and a plane parallel to the Y axis passing through the points $(-L/2, 0, 0)$ and $(w, 0, \frac{ab}{L})$.

The equation of plane # 2 is:

$$z_2 = (\frac{L}{2} - w)\ (\frac{1}{2} + \frac{x}{L})$$

$$A_2 = \int_{-\frac{L}{2}}^{w} (\frac{L}{2} - w)(\frac{1}{2} + \frac{x}{L})\ \frac{r\,dx}{\sqrt{r^2-x^2}}$$

$$A_2 = (\frac{L}{2} - w)\ \frac{r}{2}\int_{-\frac{L}{2}}^{w} \frac{dx}{\sqrt{r^2-x^2}} + (\frac{L}{2} - w)\ \frac{r}{L}\int_{-\frac{L}{2}}^{w} \frac{x\,dx}{\sqrt{r^2-x^2}}$$

$$A_2 = (\frac{L}{2} - w)\ \frac{r}{2}\ \Big/ \sin^{-1}\frac{x}{r} \Big/_{-\frac{L}{2}}^{w} + (\frac{L}{2} - w)\ \frac{r}{L}\ \Big/ -\sqrt{r^2-x^2} \Big/_{-\frac{L}{2}}^{w}$$

$$A_2 = (\frac{L}{2} - w)\ r \left[\frac{1}{2}\sin^{-1}\frac{w}{r} - \frac{1}{2}\sin^{-1}\frac{-L}{2r} - \frac{1}{L}\sqrt{r^2-w^2} + \frac{1}{L}\sqrt{r^2-\frac{L^2}{4}}\right]$$

The area of the unit load and vertical reaction $\frac{M}{EI}$ diagram, called A_p is equal to A_1 plus A_2:

$$A_p = A_1 + A_2$$

Substituting the above values for A_1 and A_2 and simplifying gives a formula for A_p in terms of the known contants of the arch. In attempting to simplify to this final equation note that the following identities hold true:

$$\sin^{-1}\frac{L}{2r} = -\ \sin^{-1}\frac{-L}{2r}$$

$$\sqrt{r^2-\frac{L^2}{4}} = r - h$$

$$A_p = r\left[\frac{L}{2}\sin^{-1}\frac{L}{2r} - w\sin^{-1}\frac{w}{r} - \sqrt{r^2-w^2} + \sqrt{r^2-\frac{L^2}{4}}\right] \quad (1)$$

The next area to be found is that of the $\frac{M}{EI}$ diagram for the redundant fixed end moment at the left springing. Again the calculus is employed, using a right circular cylinder as was done in finding the area A_p. Referring to Figure # 5, it will be noted that the required area is the surface area of a right circular cylinder bounded by

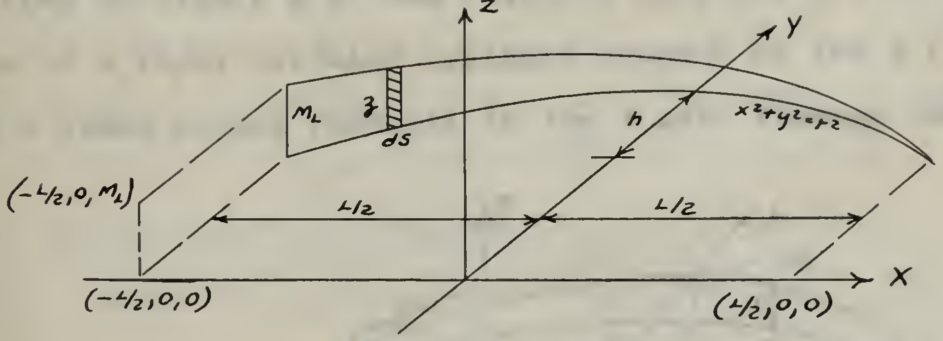


Figure #5

the X-Y plane, the plane $x = -\frac{L}{2}$, and a plane parallel to the Y axis passing through the points $(\frac{L}{2}, 0, 0)$ and $(-\frac{L}{2}, 0, M_L)$. The equation of the 2 boundary plane is:

$$z = M_L\left(\frac{1}{2} - \frac{x}{L}\right)$$

$$A_{M_L} = \int_{-\frac{L}{2}}^{\frac{L}{2}} M_L\left(\frac{1}{2} - \frac{x}{L}\right)\frac{r\,dx}{\sqrt{r^2-x^2}} = \frac{M_L r}{2}\int_{-\frac{L}{2}}^{\frac{L}{2}}\frac{dx}{\sqrt{r^2-x^2}} - \frac{M_L r}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}}\frac{x\,dx}{\sqrt{r^2-x^2}}$$

$$A_{M_L} = \frac{M_L r}{2}\Big/\sin^{-1}\frac{x}{r}\Big/_{-\frac{L}{2}}^{\frac{L}{2}} - \frac{M_L r}{L}\Big/-\sqrt{r^2-x^2}\Big/_{-\frac{L}{2}}^{\frac{L}{2}}$$

$$A_{M_L} = M_L r\left(\sin^{-1}\frac{L}{2r}\right) \quad (2)$$

Similarly, the area of the $\frac{M}{EI}$ diagram for the fixed end moment at the right springing is:

$$A_{M_R} = M_R\, r \left(\sin^{-1} \frac{L}{2r}\right) \qquad (3)$$

The last area to be found is that of the $\frac{M}{EI}$ diagram for the horizontal thrust. Once again the same right circular cylinder is used, but in this case it is necessary to pass the Z boundary plane parallel to the X axis. Referring to Figure # 6, the required area is the surface area of a right circular cylinder bounded by the X-Y plane, and a plane passed parallel to the X axis through the points

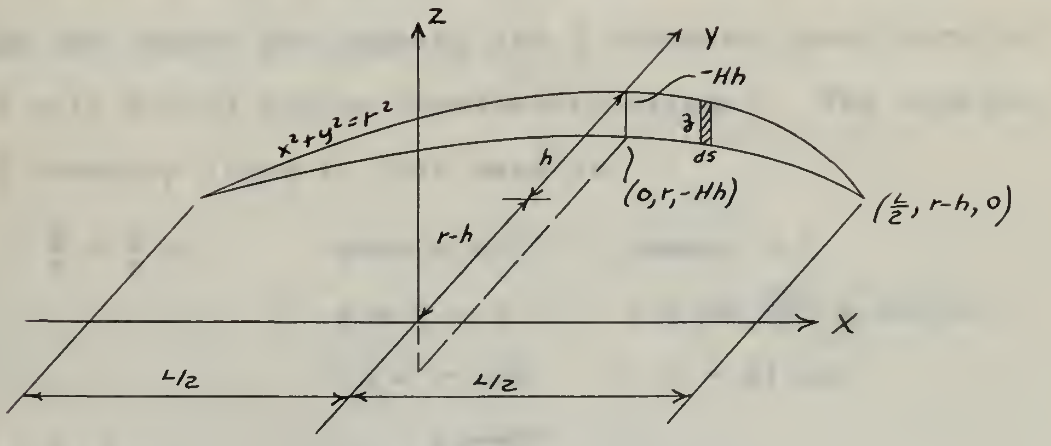


Figure # 6

$(\frac{L}{2}$, r - h, 0) and (0, r, -Hh). It is considered adviseable at this point to advance an explanation of the method of positioning the Z boundary plane in this case. In finding the area of previous $\frac{M}{EI}$ diagrams, the passing of the Z boundary plane parallel to the Y axis should be obvious after

referring to the moment diagram sketches in Figure # 2. However, in this case the reason for passing the **Z** boundary plane parallel to the X axis is not so obvious at first inspection. Referring to Figure # 2, it will be noted that the moment diagram for the horizontal thrust, when projected onto the X axis, is in the form of a circular segment. It may be difficult for the reader to understand how a plane surface parallel to the X axis could cut out such a curve when intersecting a circular cylinder. However, if the reader will consider the projection of the moment diagram onto the Y axis, such a diagram will be bounded by a straight line; and the reason for passing the **Z** boundary plane parallel to the X axis should become immediately evident. The equation of the Z boundary plane in this case is:

$$\begin{array}{lll} \frac{y}{b} + \frac{z}{c} = 1 & \text{when } z = 0 & \text{when } y = 0 \\ & y = r - h & z = Hh\,\frac{r-h}{h} = h(r-h) \\ & \therefore b = r - h & \therefore c = H(r-h) \end{array}$$

$$z = H\,(r-h-y) \qquad y = \sqrt{r^2-x^2}$$

$$z = H\,(r-h-\sqrt{r^2-x^2})$$

$$A_H = \int_{-\frac{L}{2}}^{\frac{L}{2}} z\,ds = \int_{-\frac{L}{2}}^{\frac{L}{2}} H\left\{r-h-\sqrt{r^2-x^2}\right\}\frac{r\,dx}{\sqrt{r^2-x^2}} = (Hr^2-Hhr)\int_{-\frac{L}{2}}^{\frac{L}{2}}\frac{dx}{\sqrt{r^2-x^2}} - Hr\int_{-\frac{L}{2}}^{\frac{L}{2}} dx$$

$$A_H = (Hr^2-Hhr)\,\Big/\sin^{-1}\frac{x}{r}\Big/_{-\frac{L}{2}}^{\frac{L}{2}} - Hr\,\Big/x\Big/_{-\frac{L}{2}}^{\frac{L}{2}} = (Hr^2-Hhr)2\sin^{-1}\frac{L}{2r} - HrL$$

$$A_H = Hr\left[(r-h)\,2\sin^{-1}\frac{L}{2r} - L\right] \qquad (4)$$

The areas of all the $\frac{M}{EI}$ diagrams have now been found, and the next step is to determine the centroid locations of each area.

UNIT LOAD AND VERTICAL REACTION DIAGRAM. For this $\frac{M}{EI}$ diagram, let "d" equal the distance along the X axis from the Y axis to the centroid of the area shown in Figure # 4. The $d = \bar{x}_p = \frac{M_{pY}}{A_p}$ where M_{pY} is the first moment of the area about the Y axis. As in determining A_p, it is convenient to determine the moment of areas # 1 and # 2 separately. Referring to Figure # 4 and using the accompanying information:

$$d\,M_{pY} = \int z_1\, x\, ds + \int z_2\, x\, ds \qquad z_1 = \left(\frac{L}{2} + w\right)\left(\frac{1}{2} - \frac{x}{L}\right)$$

$$z_2 = \left(\frac{L}{2} - w\right)\left(\frac{1}{2} + \frac{x}{L}\right)$$

$$ds = \frac{r\,dx}{\sqrt{r^2 - x^2}}$$

$$M_{pY} = \int_{w}^{\frac{L}{2}} \left(\frac{L}{2} + w\right)\left(\frac{1}{2} - \frac{x}{L}\right) \frac{r\,x\,dx}{\sqrt{r^2 - x^2}} + \int_{-\frac{L}{2}}^{w} \left(\frac{L}{2} - w\right)\left(\frac{1}{2} + \frac{x}{L}\right) \frac{r\,x\,dx}{\sqrt{r^2 - x^2}}$$

$$M_{pY} = \left(\frac{L}{2} + w\right) \frac{r}{2} \int_{w}^{\frac{L}{2}} \frac{x\,dx}{\sqrt{r^2 - x^2}} - \left(\frac{L}{2} + w\right) \frac{r}{L} \int_{w}^{\frac{L}{2}} \frac{x^2\,dx}{\sqrt{r^2 - x^2}} + \left(\frac{L}{2} - w\right) \frac{r}{2}$$

$$\int_{-\frac{L}{2}}^{w} \frac{x\,dx}{\sqrt{r^2 - x^2}} + \left(\frac{L}{2} - w\right) \frac{r}{L} \int_{-\frac{L}{2}}^{w} \frac{x^2\,dx}{\sqrt{r^2 - x^2}}$$

$$M_{p_Y} = \left(\frac{L}{2} + w\right)\frac{r}{2} \Big/ - \sqrt{r^2-x^2} \Big/_{-\frac{L}{2}}^{w} + \left(\frac{L}{2} - w\right)\frac{r}{L} \Big/ -\frac{x}{2}\sqrt{r^2-x^2}$$

$$+ \frac{r^2}{2} \sin^{-1} \frac{x}{r} \Big/_{w}^{\frac{L}{2}} + \left(\frac{L}{2} - w\right)\frac{r}{2} \Big/ - \sqrt{r^2-x^2} \Big/_{-\frac{L}{2}}^{w} + \left(\frac{L}{2} - w\right)\frac{r}{L}$$

$$\Big/ - \frac{x}{2} \sqrt{r^2-x^2} + \frac{r^2}{2} \sin^{-1} \frac{x}{r} \Big/_{-\frac{L}{2}}^{w}$$

Substituting limits and simplifying:

$$M_{p_Y} = \frac{wr}{2} \left[\sqrt{r^2-w^2} - \sqrt{r^2-\frac{L^2}{4}}\right] + \frac{r^3}{2} \sin^{-1} \frac{w}{r} - \frac{wr^2}{L} \sin^{-1} \frac{L}{2r} \qquad (5)$$

$$d = \bar{x}_p = \frac{M_{p_Y}}{A_p} = \frac{\text{Equation (5)}}{\text{Equation (1)}} \qquad (6)$$

To find "f" the $\bar{y}_p$ for the unit load and vertical reaction diagram.

$$M_{p_X} = \int_{w}^{\frac{L}{2}} \left(\frac{L}{2} + w\right)\left(\frac{1}{2} - \frac{x}{L}\right) \frac{r\ y\ dx}{\sqrt{r^2-x^2}} + \int_{-\frac{L}{2}}^{w} \left(\frac{L}{2} - w\right)\left(\frac{1}{2} + \frac{x}{L}\right) \frac{r\ y\ dx}{\sqrt{r^2-x^2}} \Bigg\} y = r^2-x^2$$

$$M_{p_X} = \left(\frac{L}{2} + w\right)\int_{w}^{\frac{L}{2}} \left(\frac{1}{2} - \frac{x}{L}\right) r\ dx + \left(\frac{L}{2} - w\right)\int_{-\frac{L}{2}}^{w} \left(\frac{1}{2} + \frac{x}{L}\right) r\ dx$$

$$M_{p_X} = \left(\frac{L}{2} + w\right)\frac{r}{2}\int_{w}^{\frac{L}{2}} dx - \left(\frac{L}{2} + w\right)\frac{r}{L}\int_{w}^{\frac{L}{2}} x\,dx + \left(\frac{L}{2} - w\right)\frac{r}{2}\int_{-\frac{L}{2}}^{w} dx + \left(\frac{L}{2} - w\right)\frac{r}{L}\int_{-\frac{L}{2}}^{w} x\,dx$$

$$M_{p_X} = \left(\frac{L}{2} + w\right)\frac{r}{2}\Big/ x \Big/_{w}^{\frac{L}{2}} - \left(\frac{L}{2} + w\right)\frac{r}{L}\Big/ \frac{x^2}{2} \Big/_{w}^{\frac{L}{2}} + \left(\frac{L}{2} - w\right)\frac{r}{2}\Big/ x \Big/_{-\frac{L}{2}}^{w} + \left(\frac{L}{2} - w\right)\frac{r}{L}\Big/ \frac{x^2}{2} \Big/_{-\frac{L}{2}}^{w}$$

Substituting limits and simplifying:

$$M_{p_X} = \frac{r}{8}\left(L^2 - 4w^2\right) \qquad (7)$$

$$f = \bar{y}_p = \frac{M_{p_X}}{A_p} = \frac{\text{Equation (7)}}{\text{Equation (1)}} \qquad (8)$$

MOMENT AT LEFT SPRINGING DIAGRAM. For this $\frac{M}{EI}$ diagram, let "c" equal $\bar{x}$ distance of diagram. Referring to Figure # 5 and proceeding as before:

$$M_{M_{L_Y}} = \int_{-\frac{L}{2}}^{\frac{L}{2}} z\ x\ ds = \int_{-\frac{L}{2}}^{\frac{L}{2}} M_L\left(\frac{1}{2} - \frac{x}{L}\right) \frac{r\ x\ dx}{\sqrt{r^2-x^2}} = \frac{M_L r}{2}\int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{x\ dx}{\sqrt{r^2-x^2}}$$

$$- \frac{M_L r}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{x^2\ dx}{\sqrt{r^2-x^2}}$$

$$M_{M_{L_Y}} = \frac{M_L r}{2}\Big/ -\sqrt{r^2-x^2}\Big/_{-\frac{L}{2}}^{\frac{L}{2}} - \frac{M_L r}{L}\Big/ -\frac{x}{2}\sqrt{r^2-x^2} + \frac{r^2}{2}\sin^{-1}\frac{x}{r}\Big/_{-\frac{L}{2}}^{\frac{L}{2}}$$

Substituting limits and simplifying:

$$M_{M_{L_Y}} = M_L\left(\frac{r}{2}\sqrt{r^2-\frac{L^2}{4}} - \frac{r^3}{L}\sin^{-1}\frac{L}{2r}\right) \qquad (9)$$

$$c = \bar{x}_{M_L} = \frac{M_{M_{LY}}}{A_{M_L}} = \frac{M_L\left(\frac{r}{2}\sqrt{r^2-L^2/4} - \frac{r^3}{L}\sin^{-1}\frac{L}{2r}\right)}{M_L\ r\ \left(\sin^{-1}\frac{L}{2r}\right)}$$

$$c = \frac{\sqrt{r^2-\frac{L^2}{4}}}{2\sin^{-1}\frac{L}{2r}} - \frac{r^2}{L} \qquad (10)$$

To find "m" the $\bar{y}_{M_L}$ for the M_L diagram:

$$M_{M_{L_X}} = \int_{-\frac{L}{2}}^{\frac{L}{2}} z\ y\ ds = \int_{-\frac{L}{2}}^{\frac{L}{2}} M_L \left(\frac{1}{2} - \frac{x}{L}\right) r\ dx = \frac{M_L r}{2}\int_{-\frac{L}{2}}^{\frac{L}{2}} dx - \frac{M_L r}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} x\ dx$$

$$M_{M_{L_X}} = \frac{M_L r}{2} \Big/ x \Big/_{-\frac{L}{2}}^{\frac{L}{2}} - \frac{M_L r}{L} \Big/ \frac{x^2}{2} \Big/_{-\frac{L}{2}}^{\frac{L}{2}} = \frac{M_L L r}{2} - \frac{M_L r}{L}\left(\frac{L^2}{8} - \frac{L^2}{8}\right)$$

$$M_{M_{L_X}} = \frac{M_L r L}{2} \qquad (11)$$

$$m = \bar{y}_{M_L} = \frac{M_{M_{LX}}}{A_{M_L}} = \frac{\frac{M_L r L}{2}}{M_L r \left(\sin^{-1}\frac{L}{2r}\right)} = \frac{L}{2\sin^{-1}\frac{L}{2r}}$$

$$m = \frac{L}{2\sin^{-1}\frac{L}{2r}} \qquad (12)$$

MOMENT AT RIGHT SPRINGING DIAGRAM. In a similar manner, neglecting the sign difference for $\bar{x}$, the equations for the centroid location of the $\frac{M}{EI}$ diagram for the springing moment at the right end are the same as those for the left end.

HORIZONTAL THRUST DIAGRAM. For this $\frac{M}{EI}$ diagram let "n" be the $\bar{y}$ distance of the area. Referring to Figure # 6 and proceeding as before:

$$M_{H_X} = \int_{-\frac{L}{2}}^{\frac{L}{2}} y\ z\ ds = \int_{-\frac{L}{2}}^{\frac{L}{2}} H(r-h-\sqrt{r^2-x^2})r\ dx = Hr^2\int_{-\frac{L}{2}}^{\frac{L}{2}} dx$$

$$- Hhr\int_{-\frac{L}{2}}^{\frac{L}{2}} dx - Hr\int_{-\frac{L}{2}}^{\frac{L}{2}} \sqrt{r^2-x^2}\ dx$$

$$M_{H_X} = (Hr^2 - Hhr)\int_{-\frac{L}{2}}^{\frac{L}{2}} dx - Hr\int_{-\frac{L}{2}}^{\frac{L}{2}}\sqrt{r^2-x^2}\ dx = Hr\ (r-h) \Big/ x \Big/_{-\frac{L}{2}}^{\frac{L}{2}}$$

$$- Hr \Big/ \frac{1}{2}\ (x\sqrt{r^2-x^2} + r^2\ \sin^{-1}\frac{x}{r}) \Big/_{-\frac{L}{2}}^{\frac{L}{2}}$$

Substituting limits and simplifying:

$$M_{H_X} = Hr\ (\frac{L}{2}\sqrt{r^2-\frac{L^2}{4}} - r^2\ \sin^{-1}\frac{L}{2r}) \qquad (13)$$

$$n = \bar{y}_H = \frac{M_{H_X}}{A_H} = \frac{\text{Equation (13)}}{\text{Equation (4)}} \qquad (14)$$

Since the horizontal thrust $\frac{M}{EI}$ diagram is symmetrical about the Y axis, $\bar{x}$ will obviously be zero.

Before proceeding to the third and final step in derivation of the formulae required for solution of the symmetrical circular arch of constant moment of inertia, a resumé of centroid location nomenclature is presented for convenience.

Symbol	Quantity	Equation No.
d	$\bar{x}_p$	(6)
f	$\bar{y}_p$	(8)
c	$\bar{x}_{M_L}$ or $\bar{x}_{M_R}$	(10)
m	$\bar{y}_{M_L}$ or $\bar{y}_{M_R}$	(12)
n	$\bar{y}_H$	(14)

Referring to the discussion on Pages 5, 6, 7, and 8 concerning the conjugate structure principle and having obtained the areas and centroid locations of the various component $\frac{M}{EI}$ diagrams, it is now necessary to place these $\frac{M}{EI}$ loads on the conjugate structure and set up the conditions of equilibrium in order to solve for the redundant arch reactions. Figure # 7 shows a plan view of the conjugate structure used when the unit load is placed at the crown or to the right of the crown on the arch. The various $\frac{M}{EI}$ diagrams are represented by the load points indicated by circles, considering that each load acts at its centroid location. An "X" in the circles indicates that the particular

load acts downward, and a dot indicates that the load acts upward.

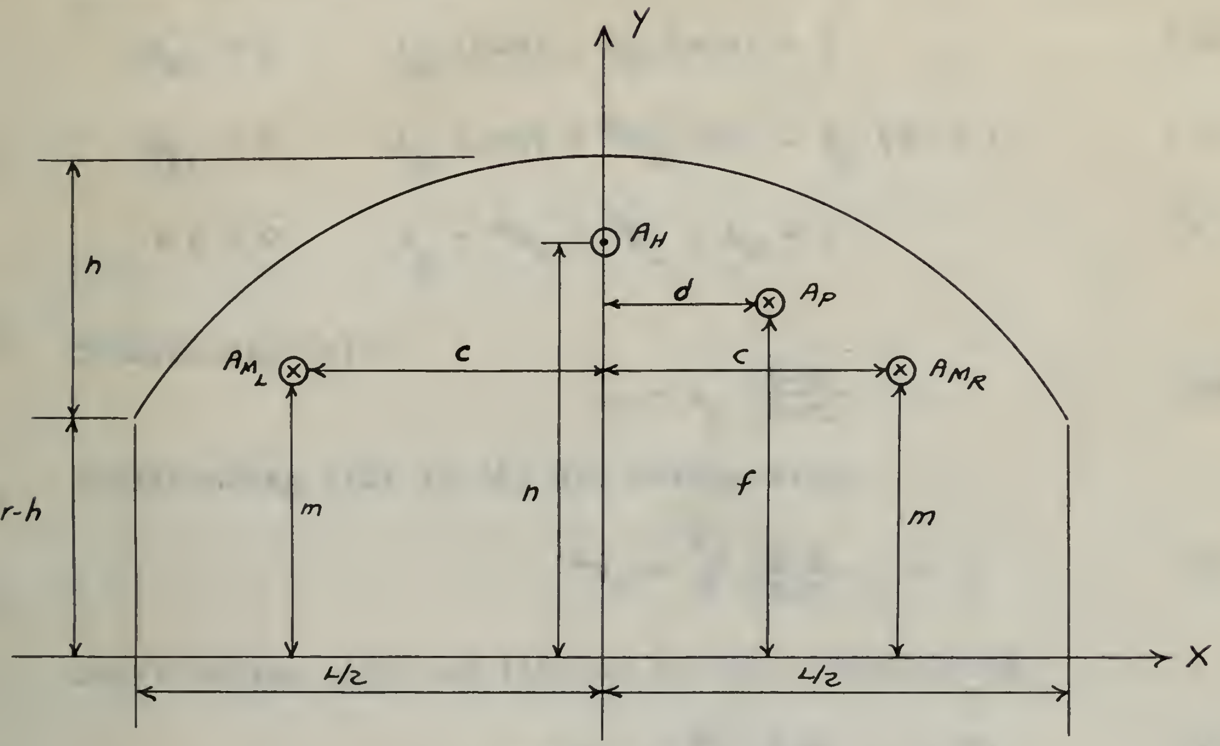


Figure # 7

As noted on Page 5, the sign convention chosen is that forces which result in tension on the inside of the real structure are assumed positive and will act downward on the conjugate structure. As an example, the loads and reactions shown in Figure # 1 are consistent with the $\frac{M}{EI}$ diagram loads acting on the conjugate structure as shown in Figure # 7. A negative answer for any reaction indicates that it is acting in a direction opposite to that shown in Figure # 1.

By taking moments about axes parallel to the X and Y axes, both passing through the centroid of ${}^{A}M_L$, the following equilibrium equations can be set up:

$$M_{XX} = 0 \qquad A_p\,(f-m) - A_H\,(n-m) = 0 \qquad \text{(a)}$$

$$M_{YY} = 0 \qquad A_p\,(c+d) + {}^{A}M_R\,(2c) - A_H\,(c) = 0 \qquad \text{(b)}$$

$$\Sigma V = 0 \qquad A_p + {}^{A}M_R + {}^{A}M_L - A_H = 0 \qquad \text{(c)}$$

Transposing (a):

$$A_H = A_p\,\frac{(f-m)}{(n-m)} \qquad (15)$$

Substituting (15) in (b) and transposing:

$${}^{A}M_R = \frac{A_p}{2}\left(\frac{f-m}{n-m} - 1 - \frac{d}{c}\right) \qquad (16)$$

Substituting (15) and (16) in (c) and simplifying:

$${}^{A}M_L = \frac{A_p}{2}\left(\frac{f-m}{n-m} - 1 + \frac{d}{c}\right) \qquad (17)$$

Equations (15), (16), and (17) are the expressions for the values of the unknown moments and thrust. Everything on the right hand side of the equations is known in terms of the arch constants as given by equations (6) through (14).
The same is true for ${}^{A}H$, ${}^{A}M_R$, and ${}^{A}M_L$, but each contains one unknown, namely H, M_R, and M_L. Equations (15), (16), and (17)

have been developed for a unit load acting at the crown or to the right of the crown; so they are VALID ONLY WHEN THE LOAD IS PLACED TO THE RIGHT OF THE CROWN OR AT THE CROWN. When the unit load is placed to the left of the crown, the right hand side of the equations remain unchanged, but ${}^{A}M_R$ and ${}^{A}M_L$ MUST BE INTERCHANGED IN EQUATIONS (16) and (17).

An example will be given herein to illustrate the procedure to be followed in the solution of a symmetrical circular arch; using a 20 foot span and a 0.8 foot rise, with the unit load placed 4.0143 feet to the right of the crown.

Prior to direct substitution of arch constants into the various formulae, it will be convenient to evaluate certain terms which appear frequently in several of the formulae.

$$w = 4.0143' \qquad L = 20.0' \qquad h = 0.80'$$

$$r = \frac{4h^2+L^2}{8h} = \frac{4(0.8)^2+(20)^2}{8\ (0.8)} = \frac{402.560}{6.40} = 62.900'$$

$$\sqrt{r^2-w^2} = \sqrt{(62.9)^2 - (4.0143)^2} = \sqrt{3940.29540} = 62.771772'$$

$$\sqrt{r^2-\frac{L^2}{4}} = r - h = 62.9 - 0.8 = 62.100'$$

$$\sin^{-1}\frac{L}{2r} = \sin^{-1}\frac{20}{125.80} = \sin^{-1} 0.1589825 = 0.1596699$$

$$\sin^{-1}\frac{w}{r} = \sin^{-1}\frac{4.0143}{62.9} = \sin^{-1} 0.06382034 = 0.06386374$$

All computations in the solution of this example are carried out to seven significant figures; however, when a number comes out evenly, the zeros necessary for seven figures are not indicated. Seven significant figures must be used to make the results of the formulae reasonably accurate, due to the sensitiveness of the formulae to extremely small variations in values. The author realizes that such restrictions make these formulae impractical for use by the designer; however, as explained in the introduction, once several different arches are solved and a set of influence lines are constructed, the required labor will be well spent.

Equations (1) through (4) will be used to find the areas of the various $\frac{M}{EI}$ diagrams.

$$A_p = r\left(\frac{L}{2}\sin^{-1}\frac{L}{2r} - w\sin^{-1}\frac{w}{r} - \sqrt{r^2-w^2} + \sqrt{r^2-\frac{L^2}{4}}\right)$$

$$A_p = 62.9\Big(10(0.1596599) - 4.0143(0.06386374) - 62.771772 + 62.100\Big)$$

$$A_p = 42.04611$$

$$A_{M_L} = M_L\, r\left(\sin^{-1}\frac{L}{2r}\right) = 62.90\, M_L\,(0.1596599) = 10.04261\, M_L$$

$$A_{M_R} = M_R\, r\left(\sin^{-1}\frac{L}{2r}\right) \qquad A_{M_R} = 10.04261\, M_R$$

$$A_H = Hr\left((r-h)\,2\sin^{-1}\frac{L}{2r} - L\right) = 62.9\,H\Big(\big(62.10\big)2\big(0.1596599\big) - 20.00\Big) = 10.707656\,H$$

To find the centroid locations of the above four areas, equations (5) through (14) will be used:

$$M_{p_y} = \frac{Wr}{2}\left(\sqrt{r^2-w^2} - \sqrt{r^2-\frac{L^2}{4}}\right) + \frac{r^3}{2} \sin^{-1}\frac{w}{r} - \frac{wr^3}{L}\sin^{-1}\frac{L}{2r} \quad (5)$$

$$M_{p_y} = \frac{4.0143(62.9)}{2}(0.671772) + \frac{(62.9)^3}{2}(0.06386374) - \frac{(4.0143)(62.9)^3}{20}(.1596599)$$

$$M_{p_y} = 56.3724$$

$$d = \bar{x}_p = \frac{M_{p_y}}{A_p} \quad (6)$$

$$d = \frac{56.3724}{42.04611} = 1.340728$$

$$M_{p_x} = \frac{r}{8}(L^2 - 4w^2) \quad (7)$$

$$M_{p_x} = \frac{62.9}{8}\left((20)^2 - 4(4.0143)^2\right) = 2638.19567$$

$$f = \bar{y}_p = \frac{M_{p_x}}{A_p} \quad (8)$$

$$f = \frac{2638.19567}{42.04611} = 62.74529$$

$$c = \frac{\sqrt{r^2-L^2/4}}{2\sin^{-1}\frac{L}{2r}} - \frac{r^2}{L} \quad (10)$$

$$c = \frac{62.10}{2(.1596599)} - \frac{(62.90)^2}{20} = -3.3447$$

$$m = \frac{L}{2 \sin^{-1} \frac{L}{2r}} \qquad (12)$$

$$m = \frac{20}{2(.1596599)} = 62.63311$$

$$M_{H_x} = Hr \left(\frac{L}{2}\sqrt{r^2-\frac{L^2}{4}} - r^2 \sin^{-1} \frac{L}{2r}\right) \qquad (13)$$

$$M_{H_x} = 62.9\ H \left(621.0 - (62.9)^2(0.1596599)\right) = 671.78848\ H$$

$$n = \bar{y}_H = \frac{M_{H_x}}{A_H} \qquad (14)$$

$$n = \frac{671.78848H}{10.707656H} = 62.739079$$

Knowing the values of the various areas and centroid distances, the last step is to substitute these values in equations (15), (16), and (17).

$$A_H = A_p \frac{f-m}{n-m} \qquad (15)$$

$$10.707656H = 42.04611 \frac{62.74529 - 62.63311}{62.739079- 62.63311} \qquad H = 4.1568^{k}$$

$$A_{M_R} = \frac{A_p}{2} \left(\frac{f-m}{n-m} - 1 - \frac{d}{c}\right) \qquad (16)$$

$$10.04261\ M_R = \frac{42.04611}{2} \left(\frac{62.74529 - 62.63311}{62.739079- 62.63311} - 1 - \frac{1.340728}{3.3447}\right)$$

$$M_R = -\ 0.7165^{ft.kips}$$

$$A_{M_L} = \frac{A_p}{2} \left(\frac{f-m}{n-m} - 1 + \frac{d}{c}\right) \qquad (17)$$

$$10.04261\ M_L = \frac{42.04611}{2} \left(\frac{62.74529 - 62.63311}{62.73908 - 62.63311} - 1 + \frac{1.340728}{3.3447}\right)$$

$$M_L = 0.9618^{\ ft.kips}$$

It should be noted that M_R is negative, indicating that this moment tends to cause compression on the inside of the real structure.

If the unit load had been placed 4.0143 feet to the left of the crown, "H" is the same as above, but the values of ${}^{A}M_R$ and ${}^{A}M_L$ must be interchanged. Accordingly for a unit load in this new position, the redundant values are:

$H = 4.1568^{k}$ $M_R = 0.9618^{k}$ $M_L = -\ 0.7155^{'k}$

To find the values of H, M_R and M_L for the remaining points on the span, the same procedure is followed, substituting the proper value of "w" in each case. However, for each new load point only one of the areas and its centroid distances must be recomputed, in addition to the three final euqations. If it is desired to find the influence lines by using ten points, as is the usual case, the above procedure is required for only five of the points, since values for symmetrical points can be found by interchanging M_R and M_L as was shown in the example.

This completes the derivation and application of the exact method of solution for circular arches whose cross sectional moment of inertia is constant.

RELATIONSHIPS BETWEEN ARCHES HAVING THE SAME RATIO OF RISE TO SPAN

For arches having the same rise to span ratio, certain relationships will be developed which will save considerable time in the solution of a series of arches. Using the equations of PART I of this thesis, and keeping the ratio of rise to span constant, the following simplified formulae are derived:

Let $K = \frac{h}{L}$, the rise to span ratio, which is now constant.

Let $v = \frac{w}{L}$, where "v" is a constant indicating the position of the load point off the center of span.

$$r = \frac{4h^2+L^2}{8h} = \frac{h}{2} + \frac{L^2}{8h} = \frac{KL}{2} + \frac{L}{8K} = L\left(\frac{K}{2} + \frac{1}{8k}\right) = C_1 L$$

$$\sin^{-1}\frac{L}{2r} = \sin^{-1}\frac{L}{2C_1 L} = \sin^{-1}\frac{1}{2C_1} = C_2$$

$$\sin^{-1}\frac{w}{r} = \sin^{-1}\frac{vL}{C_1 L} = \sin^{-1}\frac{v}{C_1} = C_3$$

$$\sqrt{r^2-w^2} = \sqrt{C_1^2 L^2 - v^2 L^2} = L\sqrt{C_1^2 - v^2} = C_4 L$$

$$\sqrt{r^2-\frac{L^2}{4}} = \sqrt{C_1^2 L^2 - \frac{L^2}{4}} = L\sqrt{C_1^2 - \frac{1}{4}} = C_5 L$$

The values of the arbitrary constants C_1 through C_5 could be computed if desired, however the only point of interest to be noted is that they are constant values for the same load point on arches having the same rise to span ratio.

All constants used in the following discussion will be noted as C_6, C_7, etc.

The above values are now substituted as follows:

In equation (1) $A_p = C_1 L \left(\frac{L}{2} C_2 - vLC_3 - C_4 L + C_5 L\right)$

$$= C_1 L^2 \left(\frac{C_2}{2} - vC_3 - C_4 + C_5\right) = C_6 L^2$$

In equation (2) $A_{M_L} = M_L \; C_1 \; L \; C_2 = C_7 \; M_L L$

In equation (3) $A_{M_R} = M_R \; C_1 \; L \; C_2 = C_7 \; M_R \; L$

In equation (4) $A_H = C_1 L \; H \; (2C_5 \; L \; C_2 - L) = H \; L^2 \; C_1 (2C_5 C_2 - 1)$

$$A_H = C_8 \; H \; L^2$$

In equation (5) $M_{P_Y} = \frac{vLc1L}{2} (C_4 L - C_5 L) + \frac{C_1{}^3 L^3}{2} \; C_3$

$$- \frac{vLC_1{}^3 L^3}{L} \; C_2 = C_9 \; L^3$$

In equation (6) $d = \frac{C_9 \; L^3}{C_6 \; L^2} = C_{10} \; L$

In equation (7) $M_{P_X} = \frac{C_1 L}{8} (L^2 - 4v^2 L^2) = L^3 \frac{C_1}{8} (1 - 4v^2) = C_{11} L^3$

In equation (8) $f = \frac{C_{11} L^3}{C_6 L^2} = C_{12} L$

In equation (10) $C = \frac{C_5 L}{2C_2} - \frac{C_1{}^2 L^2}{L} = C_{13} L$

In equation (12) $m = \frac{L}{2C_2} = C_{14} L$

In equation (13) $M_{H_X} = HC_1 L\ (\frac{L}{2}\ C_5 L - C_1{}^2 L^2 C_2) = C_{15} HL^3$

In equation (14) $n = \frac{C_{15} HL^3}{C_8\ HL^2} = C_{16} L$

Now applying the above simplifications in the three final equations, the following results are obtained:

$A_H = A_p \frac{f-m}{n-m}$ $\qquad C_8 HL^2 = C_6 L^2\ \frac{C_{12} L - C_{14} L}{C_{16} L - C_{14} L}$

$$H = \frac{C_6}{C_8}\ (\frac{C_{12} - C_{14}}{C_{16} - C_{14}}) = C_{17} \qquad H = C_{17} \qquad (18)$$

$A_{M_R} = \frac{A_p}{2}\ (\frac{f-m}{n-m} - 1 - \frac{d}{c})$ $\qquad C_7 M_R L = \frac{C_6 L^2}{2}\ (\frac{C_{12} L - C_{14} L}{C_{16} L - C_{14} L} - 1 - \frac{C_{10} L}{C_{13} L}$

$$M_R = L\ \frac{C_6}{C_7 2}\ (\frac{C_{12} - C_{14}}{C_{16} - C_{14}} - 1 - \frac{C_{10}}{C_{13}})$$

$$M_R = C_{18} L \qquad (19)$$

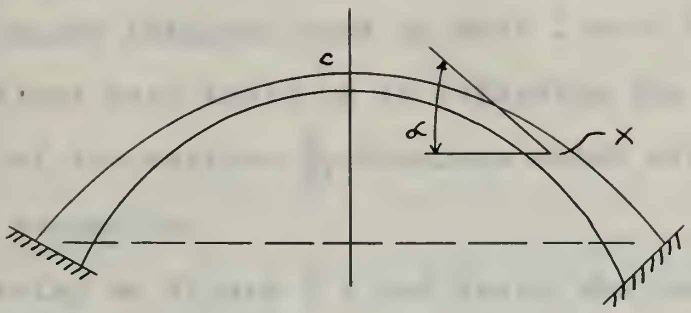
$A_{M_L} = \frac{A_p}{2}\ (\frac{f-m}{n-m} - 1 + \frac{d}{c})$ Obviously $M_L = C_{19} L$ $\qquad M_L = C_{19} L \qquad (20)$

Equations (18), (19), and (20) bring out the important conclusions that on arches having the same rise to span ratio, for the same load point, (1) THE VALUE OF THE HORIZONTAL THRUST REMAINS CONSTANT, AND (2) THE VALUE OF THE SPRINGING MOMENTS VARIES DIRECTLY AS THE LENGTH OF SPAN. For example, if the moment at the right springing on an arch of 10' span and having a rise to span ratio of 0.1 is, say, 0.555; then the moment at the right springing on an arch of 20' span with a rise to span ratio of 0.1 is $\frac{20}{10}$ (0.555) or 1.110, providing the same load is acting at the same load point.

PART II

VARYING MOMENT OF INERTIA

In this part exact formulae will be developed for the solution of symmetrical circular arches which have their moment of inertia varying as a specific function. The procedure and principles used will be the same as utilized in part one of this thesis. Figure # 8 below shows that the moment of inertia at any section "X" varies as the moment of inertia at the crown times the secant of the angle "alpha".



$I_X = I_C \sec\alpha$

Figure # 8

The modulus of elasticity, E, will again be dropped from all calculations, since it appears in each formula as a multiplier and consequently cancels out. However, the moment of inertia must be included in calculation, since the inertia is now a variable quantity.

To derive the expression for the moment of inertia at any point, the procedure is as follows:

$I_X = I_C \sec \alpha = I_C \frac{ds}{dx}$ since for any differential arch segment the secant of alpha will equal $\frac{ds}{dx}$.

The ordinates of the $\frac{M}{EI}$ diagrams are exactly the same as those for the arch with constant moment of inertia in PART I. Accordingly the equations of the Z boundary planes that were passed through the right circular cylinder in PART I remain unchanged. The only change in this part is that each basic integral used in PART I must be divided by I_X.

The first step again is to determine the equations for the areas of the various $\frac{M}{EI}$ diagrams which will act on the conjugate structure.

Referring to Figure # 4 and using the same terminology that appears there, the equation for A_p is:

$$A_p = \int_{w}^{\frac{L}{2}} \left(\frac{L}{2} + w\right)\left(\frac{1}{2} - \frac{x}{L}\right) \frac{ds}{I_X} + \int_{-\frac{L}{2}}^{w} \left(\frac{L}{2} - w\right)\left(\frac{1}{2} + \frac{x}{L}\right) \frac{ds}{dx}$$

$$A_p = \left(\frac{L}{2} + w\right) \int_{w}^{\frac{L}{2}} \left(\frac{1}{2} - \frac{x}{L}\right) \frac{dx}{I_c} + \left(\frac{L}{2} - w\right) \int_{-\frac{L}{2}}^{w} \left(\frac{1}{2} + \frac{x}{L}\right) \frac{dx}{I_c}$$

$$A_p = \left(\frac{L}{2} + w\right) \frac{1}{2I_c} \int_{w}^{\frac{L}{2}} dx - \left(\frac{L}{2} + w\right) \frac{1}{LI_c} \int_{w}^{\frac{L}{2}} x\,dx + \left(\frac{L}{2} - w\right) \frac{1}{2I_c} \int_{-\frac{L}{2}}^{w} dx$$

$$+ \left(\frac{L}{2} - w\right) \frac{1}{LI_c} \int_{-\frac{L}{2}}^{w} x\,dx$$

$$A_p = \left(\frac{L}{2} + w\right) \frac{1}{2I_c} \left(\frac{L}{2} - w\right) - \left(\frac{L}{2} + w\right) \frac{1}{LI_c} \left(\frac{L^2}{8} - \frac{w^2}{2}\right) + \left(\frac{L}{2} - w\right) \frac{1}{2Ic} \left(w + \frac{L}{2}\right)$$

$$+ \left(\frac{L}{2} - w\right) \frac{1}{LI_c} \left(\frac{w^2}{2} - \frac{L^2}{8}\right)$$

Simplifying the above expression:

$$A_p = \frac{1}{2I_c} \left(\frac{L^2}{4} - w^2\right) \qquad (21)$$

Referring to Figure # 5 and using the basic integral established there, the area for the $\frac{M}{EI}$ diagram of the left hand springing moment will now be found.

$$A_{M_L} = \int_{-\frac{L}{2}}^{\frac{L}{2}} M_L \left(\frac{1}{2} - \frac{x}{L}\right) \frac{ds}{I_x} = M_L \int_{-\frac{L}{2}}^{\frac{L}{2}} \left(\frac{1}{2} - \frac{x}{L}\right) \frac{dx}{I_c} = \frac{M_L}{I_c} \int_{-\frac{L}{2}}^{\frac{L}{2}} \left(\frac{1}{2} - \frac{x}{L}\right) dx$$

$$A_{M_L} = \frac{M_L}{2I_c} \int_{-\frac{L}{2}}^{\frac{L}{2}} dx - \frac{M_L}{LI_c} \int_{-\frac{L}{2}}^{\frac{L}{2}} x\,dx = \frac{M_L L}{2I_c} \qquad A_{M_L} = \frac{M_L L}{2I_c} \quad (22)$$

Similarly, the area of the $\frac{M}{EI}$ diagram for the moment at the right hand springing is: $A_{M_R} = \frac{M_R L}{2I_c}$ (23)

Referring to Figure # 6 and using the basic integral established there, the area of the $\frac{M}{EI}$ diagram for the horizontal thrust is:

$$A_H = \int_{-\frac{L}{2}}^{\frac{L}{2}} (r - h - \sqrt{r^2-x^2}) \frac{ds}{I_x} = \int_{-\frac{L}{2}}^{\frac{L}{2}} H (r - h - \sqrt{r^2-x^2}) \frac{dx}{I_c}$$

$$A_H = \frac{Hr}{I_c}\int_{-\frac{L}{2}}^{\frac{L}{2}} dx - \frac{Hh}{I_c}\int_{-\frac{L}{2}}^{\frac{L}{2}} dx - \frac{H}{I_c}\int_{-\frac{L}{2}}^{\frac{L}{2}} \sqrt{r^2-x^2}\, dx = (\frac{Hr}{I_c} - \frac{Hh}{I_c})\int_{-\frac{L}{2}}^{\frac{L}{2}} dx$$

$$- \frac{H}{I_c}\int_{-\frac{L}{2}}^{\frac{L}{2}} \sqrt{r^2-x^2}\, dx$$

$$A_H = \frac{HL}{I_c}(r - h) - \frac{H}{I_c}\left/ \frac{1}{2}(x\sqrt{r^2-x^2} + r^2 \sin^{-1}\frac{x}{r}\right/_{-\frac{L}{2}}^{\frac{L}{2}}$$

Substituting limits and simplifying:

$$A_H = \frac{H}{I_c}\left[\frac{L}{2}(r-h) - r^2 \sin^{-1}\frac{L}{2r}\right] \quad (24)$$

Having obtained the areas of the $\frac{M}{EI}$ diagrams, the next step is to determine the centroid locations of these areas.

UNIT LOAD AND VERTICAL REACTION DIAGRAM. For this diagram, as in PART I, let $d = \bar{x}_p$ and $f = \bar{y}_p$.

$$M_{p_y} = \int_{w}^{\frac{L}{2}} \left(\frac{L}{2} + w\right)\left(\frac{1}{2} - \frac{x}{L}\right) \frac{x\,dx}{I_c} + \int_{-\frac{L}{2}}^{w} \left(\frac{L}{2} - w\right)\left(\frac{1}{2} + \frac{x}{L}\right) \frac{x\,dx}{I_c}$$

Integrating, substituting limits, and simplifying, the result is:

$$M_{p_y} = \frac{w}{6I_c}\left(\frac{L^2}{4} - w^2\right) \qquad (25\text{-a})$$

$$d = \bar{x}_p = \frac{M_{p_y}}{A_p} = \frac{\frac{w}{6I_c}\left(\frac{L^2}{4} - w^2\right)}{\frac{1}{2I_c}\left(\frac{L^2}{4} - w^2\right)} = \frac{w}{3} \qquad d = \frac{w}{3} \qquad (25)$$

$$M_{p_x} = \int_{w}^{\frac{L}{2}} \left(\frac{L}{2} + w\right)\left(\frac{1}{2} - \frac{x}{L}\right) \frac{y\,dx}{I_c} + \int_{-\frac{L}{2}}^{w} \left(\frac{L}{2} - w\right)\left(\frac{1}{2} + \frac{x}{L}\right) \frac{y\,dx}{I_c}$$

$$M_{p_x} = \left(\frac{L}{2} + w\right)\int_{w}^{\frac{L}{2}} \left(\frac{1}{2} - \frac{x}{L}\right) \frac{\sqrt{r^2 - x^2}\,dx}{I_c} + \left(\frac{L}{2} - w\right)\int_{-\frac{L}{2}}^{w} \left(\frac{1}{2} + \frac{x}{L}\right) \frac{\sqrt{r^2 - x^2}\,dx}{I_c}$$

Integrating, substituting limits, and simplifying, the result is:

$$M_{p_x} = \frac{L}{4I_c}\left(\frac{L}{2}(r-h) + r^2 \sin^{-1}\frac{L}{2r}\right) - \frac{w}{2I_c}\left(w\sqrt{r^2-w^2} + r^2 \sin^{-1}\frac{w}{r}\right)$$

$$+ \frac{1}{3I_c}\left\{\left(r^2-\frac{L^2}{4}\right)^{\frac{3}{2}} - \left(r^2-w^2\right)^{\frac{3}{2}}\right] \qquad (26a)$$

$$f = \bar{y}_p = \frac{M_{p_x}}{A_p} = \frac{\text{Equation (26a)}}{\text{Equation (21)}} \qquad (26)$$

MOMENT AT THE SPRINGINGS. For these diagrams, as in PART I, let $c = \bar{x}_{M_L}$ or $\bar{x}_{M_R}$ and $m = \bar{y}_{M_L}$ or $\bar{y}_{M_R}$.

$$M_{M_{L_Y}} = \int_{-\frac{L}{2}}^{\frac{L}{2}} M_L\left(\frac{1}{2} - \frac{x}{L}\right)\frac{x\,dx}{I_c} = \frac{M_L}{2I_c}\int_{-\frac{L}{2}}^{\frac{L}{2}} x\,dx - \frac{M_L}{LI_c}\int_{-\frac{L}{2}}^{\frac{L}{2}} x^2dx$$

$$M_{M_{L_Y}} = \frac{M_L}{2I_c}\Big/\frac{x^2}{2}\Big/_{-\frac{L}{2}}^{\frac{L}{2}} - \frac{M_L}{LI_c}\Big/\frac{x^3}{3}\Big/_{-\frac{L}{2}}^{\frac{L}{2}} = \frac{M_L}{LI_c}\left(\frac{L^3}{24} + \frac{L^3}{24}\right) = \frac{M_L L^2}{12I_c} \qquad (27a)$$

$$c = \bar{x}_{M_L} \text{ or } \bar{x}_{M_R} = \frac{M_{M_{LY}}}{A_{M_L}} = \frac{M_L L^2/12I_c}{M_L L/2\,I_c} = \frac{L}{6} \qquad c = \frac{L}{6} \qquad (27)$$

$$M_{M_{L_X}} = \int_{-\frac{L}{2}}^{\frac{L}{2}} M_L\left(\frac{1}{2} - \frac{x}{L}\right) \frac{y\,dx}{I_c} = \frac{M_L}{2I_c}\int_{-\frac{L}{2}}^{\frac{L}{2}} \sqrt{r^2-x^2}\;dx - \frac{M_L}{LI_c}\int_{-\frac{L}{2}}^{\frac{L}{2}} \sqrt{r^2-x^2}\;x\;dx$$

$$M_{M_{L_X}} = \frac{M_L}{2I_c}\Big/ \frac{1}{2}\left(x\sqrt{r^2-x^2} + r^2 \sin^{-1}\frac{x}{r}\right)\Big/_{-\frac{L}{2}}^{\frac{L}{2}} - \frac{M_L}{LI_c}\Big/ -\frac{1}{3}\left(r^2-x^2\right)^{\frac{3}{2}}\Big/_{-\frac{L}{2}}^{\frac{L}{2}}$$

Substituting and simplifying:

$$M_{M_{L_X}} = \frac{M_L}{I_c}\left(\frac{L}{4}\sqrt{r^2-\frac{L^2}{4}} + \frac{r^2}{2}\sin^{-1}\frac{L}{2r}\right) \qquad (28a)$$

$$m = \bar{y}_{M_L} \text{ or } \bar{y}_{M_R} = \frac{M_{M_{L_X}}}{A_{M_L}} = \frac{M_L/I_c\left(\frac{L}{4}\sqrt{r^2-L^2/4} + r^2/2\,\sin^{-1}\frac{L}{2r}\right)}{M_L\,L/2I_c}$$

$$m = \frac{1}{2}\sqrt{r^2-\frac{L^2}{4}} + \frac{r^2}{L}\sin^{-1}\frac{L}{2r} \qquad m = \frac{r-h}{2} + \frac{r^2}{L}\sin^{-1}\frac{L}{2r} \qquad (28)$$

HORIZONTAL THRUST DIAGRAM. For the $\frac{M}{EI}$ diagram of the horizontal thrust, the centroid distance along the X axis is 0, and the centroid distance along the Y axis is "n".

$$M_{H_X} = \int_{-\frac{L}{2}}^{\frac{L}{2}} y\ z\ ds = \int_{-\frac{L}{2}}^{\frac{L}{2}} H(r-h-\sqrt{r^2-x^2})\sqrt{r^2-x^2}\ \frac{dx}{I_c}$$

$$= \frac{H}{I_c}\int_{-\frac{L}{2}}^{\frac{L}{2}} (r-h-\sqrt{r^2-x^2})\ \sqrt{r^2-x^2}\ dx$$

$$M_{H_X} = \frac{Hr}{I_c}\int_{-\frac{L}{2}}^{\frac{L}{2}} \sqrt{r^2-x^2}\ dx - \frac{Hh}{I_c}\int_{-\frac{L}{2}}^{\frac{L}{2}} \sqrt{r^2-x^2}\ dx - \frac{H}{I_c}\int_{-\frac{L}{2}}^{\frac{L}{2}} r^2\ dx$$

$$+ \frac{H}{I_c}\int_{-\frac{L}{2}}^{\frac{L}{2}} x^2\ dx$$

$$M_{H_X} = \left(\frac{Hr}{I_c} - \frac{Hh}{I_c}\right) \Big/ \frac{1}{2}\left(x\sqrt{r^2-x^2} + r^2\sin^{-1}\frac{x}{r}\right)\Big/_{-\frac{L}{2}}^{\frac{L}{2}} - \frac{Hr^2L}{I_c} + \frac{H}{I_c}\Big/\frac{x^3}{3}\Big/_{-\frac{L}{2}}^{\frac{L}{2}}$$

Substituting limits and simplifying:

$$M_{H_X} = \frac{H}{I_c}\left[(r-h)\left(\frac{L}{2}\sqrt{r^2-\frac{L^2}{4}} + r^2\sin^{-1}\frac{L}{2r}\right) - r^2L + \frac{L^3}{12}\right] \qquad (29a)$$

$$n = \bar{y}_{A_H} = \frac{M_{H_X}}{A_H} = \frac{\text{Equation (29a)}}{\text{Equation (24)}} \qquad (29)$$

The final step in the development of this method is the substitution of the values for the various $\frac{M}{EI}$ diagram areas and their centroid distances in the equilibrium equations that result from the conjugate structure. Since the solution of the conjugate structure equations is the same whether the moment of inertia is constant or varying, equations (15), (16), and (17) of PART ONE may again be used. The illustrative example given in PART ONE, to illustrate the application of formulae to the solution of an arch of constant moment of inertia, may also be used as a guide in the use of the formula of this part in the solution of arches with variable moment of inertia.

RELATIONSHIPS BETWEEN ARCHES HAVING THE SAME RATIO OF RISE TO SPAN

For arches having the same rise to span ratio, certain relationships will be developed for the solution of a series of arches whose cross sectional moment of inertia varies. Using the equations of PART II of this thesis, and keeping the ratio of rise to span constant, the following simplified formulae result:

As for the arch with constant inertia, refer to Pg. 29.

$$k = \frac{n}{L}, \quad v = \frac{w}{L}, \quad r = C_1L, \quad \sin^{-1}\frac{L}{2r} = C_2$$

$$\sin^{-1}\frac{w}{r} = C_3, \quad \sqrt{r^2-w^2} = C_4L, \quad \sqrt{r^2-\frac{L^2}{4}} = C_5L$$

The above values are now substituted as follows:

In equation (21) $A_p = \frac{1}{2I_c}\left(\frac{L^2}{4} - v^2L^2\right) = K_1L^2$

In equation (22) $A_{M_L} = \frac{M_L L}{2I_c} = K_2M_LL$

In equation (23) $A_{M_R} = K_2M_RL$

In equation (24) $A_H = \frac{H}{I_c}\left(\frac{L}{2}(C_1L - KL) - C_1^2L^2C_2\right)$

$$A_H = HL^2\frac{1}{I_c}\left(\frac{1}{2}(C_1-K) - C_1^2C_2\right) = K_3HL^2$$

In equation (25) $d = \frac{w}{3} = \frac{vL}{3} = K_4L$

In equation (26a)

$$M_{P_X} = \frac{L}{4I_c} \left(\frac{L}{2} (C_1L - KL) + C_1^2L^2C_2\right) - \frac{vL}{2I_c} \left(vL (C_4L) + C_1^2L^2C_3\right)$$

$$+ \frac{1}{3I_c} (C_5^3 L^3 - C_4^3 L^3) = K_5L^3$$

In equation (26) $f = \frac{M_{P_X}}{A_p} = \frac{K_5L^3}{K_1L^2} = K_6L$

In equation (27) $c = \frac{L}{6} = K_7L$

In equation (28) $m = \frac{C_5L}{2} + \frac{C_1^2L^2}{L} C_2 = K_8L$

In equation (29a)

$$M_{h_X} = \frac{H}{I_c} \left((C_5L)\left(\frac{L}{2} C_5L + C_1^2L^2C_2\right) - C_1^2 L^2 L + \frac{L^3}{12} \right) = K_9L^3h$$

In equation (29) $n = \frac{M_{H_X}}{A_H} = \frac{K_9L^3H}{K_3L^2H} = K_{10}L$

Applying the above simplifications in equations (15), (16), and (17), the following results are obtained:

$$A_H = A_p \frac{f-m}{n-m} \qquad K_3HL^2 = K_1L^2 \quad \frac{K_6L - K_8L}{K_{10}L - K_8L}$$

$$H = \frac{K_1}{K_3} \left(\frac{K_6 - K_8}{K_{10} - K_8}\right) = K_{11} \qquad H = K_{11} \qquad (30)$$

$$A_{M_R} = \frac{A_p}{2} \left(\frac{f-m}{n-m} - 1 - \frac{d}{c}\right) \qquad K_2M_RL = \frac{K_1L^2}{2} \left(\frac{K_6 - K_8}{K_{10} - K_8} - 1 - \frac{K_4}{K_7}\right)$$

$$M_R = K_{12} L \qquad (31)$$

$$A_{M_L} = \frac{A_p}{2} \left(\frac{f-m}{n-m} - 1 + \frac{d}{c}\right)$$ Similarly to M_R

$$M_L = K_{13}L \qquad (32)$$

Equations (30), (31), and (32) bring out the important results, for a symmetrical circular arch with a cross sectional moment of inertia that varies as the secant function, that for a constant ratio of rise to span, with the same load point, (1) the HORIZONTAL THRUST REMAINS CONSTANT, and (2) the SPRINGING MOMENT VARIES DIRECTLY AS THE SPAN LENGTH. It is interesting to note that these results compare exactly with the results found for prismatic circular arches as shown by equations (18), (19), and (20) on Page 31.

REVIEW OF RESULTS DEVELOPED FOR PARABOLIC ARCHES

As explained in the introduction, this thesis concerning circular arches is directly analogous to a thesis prepared by Iselin and LaLande concerning parabolic arches. Prior to presenting unit redundant values for the circular arch and comparing them with unit values for the parabolic arch, it is considered adviseable to present herein the formulae and results developed by Iselin and LaLande for parabolic arches.

Using the same method of application of the conjugate structure and the same terminology, Iselin and LaLande developed the following formulae for symmetrical parabolic arches with constant moment of inertia:

$$(1)\quad A_p = \frac{L}{2k}\left(\frac{LR}{4} + \frac{4N^3}{3L} - \frac{R^3}{6L} + \frac{k^2}{8}\ \ln\left(\frac{R+L}{R-L}\right)\right) - \frac{w}{k}\left(WN + \frac{k^2}{8}\ \ln 4\left(\frac{W+N}{K}\right)^2\right)$$

$$(2)\quad A_{M_R} = \frac{M_R}{4}\left(\frac{LR}{K} + \frac{k}{2}\ \ln\left(\frac{R+L}{R-L}\right)\right)$$

$$(3)\quad A_{M_L} = \frac{M_L}{4}\left(\frac{LR}{K} + \frac{k}{2}\ \ln\left(\frac{R+L}{R-L}\right)\right)$$

$$(4)\quad A_H = \frac{H}{8}\left\{\left(\frac{1}{4} + \frac{4h}{k}\right)\left(LR + \frac{k^2}{2}\ \ln\left(\frac{R+L}{R-L}\right)\right) - \frac{LR^3}{2k^2}\right]$$

$$(5)\quad M_{p_y} = \frac{WR^3}{48k} - \frac{WN^3}{6k} - \frac{WNK}{16} + \frac{WRK}{132} - \frac{K^3}{128} \ln 4 \left(\frac{W+N}{K}\right)^2 + \frac{WK^3}{64L} \ln \left(\frac{R+L}{R-L}\right)$$

$$(5a)\quad d = \bar{x}_p = \frac{M_{p_v}}{A_p}$$

$$(6)\quad M_{p_x} = \frac{L^2R^3}{320k^2} - \frac{W^2N^3}{10k^2} - \frac{L^2R}{128} + \frac{W^2N}{16} + \frac{R^3}{120} - \frac{N^3}{15} - \frac{K^2L}{256} \ln \left(\frac{R+L}{R-L}\right)$$

$$+ \frac{WK^2}{128} \ln 4 \left(\frac{W+N}{K}\right)^2$$

$$(6a)\quad f = \bar{y}_p = \frac{M_{p_x}}{A_p}$$

$$(8)\quad c = \bar{x}_{M_L} \text{ or } \bar{x}_{M_R} = \frac{R^3}{4LR + 2K^2 \ln \left(\frac{R+L}{R-L}\right)} - \frac{K^2}{8L}$$

$$(10)\quad m = \bar{y}_{M_L} \text{ or } \bar{y}_{M_R} = \frac{L}{2k} \quad (\text{"C"})$$

$$(13)\quad M_{H_X} = \frac{H}{K} \left[\left(\frac{1}{2} + \frac{4h}{k}\right)\left(\frac{LR^3}{64} - \frac{LRK^2}{128} - \frac{K^4}{256} \ln \left(\frac{R+L}{R-L}\right)\right) - \frac{L^3R^3}{96K^2} \right]$$

$$(14)\quad n = \bar{y}_H = \frac{M_{H_X}}{A_H}$$

$$(15)\quad A_H = A_p \left(\frac{m-f}{m-n}\right)$$

$$(16)\quad A_{M_R} = \frac{A_p}{2} \left(\frac{m-f}{m-n} + \frac{d}{c} - 1\right)$$

$$(17)\quad A_{M_L} = \frac{A_p}{2} \left(\frac{m-f}{m-n} - \frac{d}{c} - 1\right)$$

When unit load is located to the left of the crown.

In the above formulae, the following substitutions were made:

$$R = \sqrt{L^2 + K^2} \qquad K = \frac{L^2}{4h}$$

$$N = \sqrt{W^2 + \frac{K^2}{4}} \qquad y = \frac{x^2}{k} \text{ (Equation of parabola)}$$

For the symmetrical parabolic arch with moment of inertia varying as the secant function, three (3) extremely simple formulae are developed, which may be solved directly for the value of the arch redundants. These simple formulae are:

$$(30)\ H = \frac{15L}{8h}\left(\frac{1}{8} - v^2 + 2v^4\right)$$

$$(31)\ M_R = \frac{L}{4}\left(\frac{1}{8} + v - 3v^2 - 4v^3 + 10v^4\right)$$

$$(32)\ M_L = \frac{L}{4}\left(\frac{1}{8} - v - 3v^2 + 4v^3 + 10v^4\right)$$

When unit load is located to left of crown.

where $v = \frac{w}{L}$

Iselin and LaLande further demonstrated that for either the prismatic or non-prismatic parabolic arch, with a constant ratio of rise to span, the HORIZONTAL THRUST REMAINS CONSTANT and the SPRINGING MOMENT VARIES DIRECTLY AS THE SPAN LENGTH. Also, it is shown that the three (3) simple formulae developed are equally applicable, within 5%, for the solution of either

prismatic or non-prismatic arches up to a rise to span ratio of 0.2.

Redundant values for unit parabolic arches of various rise to span ratios were computed by Iselin and LaLande and presented in tabular form; however, these values are not shown here since they will be included in the next section of this thesis and compared with similar results for the circular arch.

Prior to accepting the results of Iselin's and LaLande's thesis, the author, after careful review of the work, felt it necessary to prove certain relationships. These relationships were undoubtedly determined, but were not included in the write-up of the thesis; so it is considered adviseable to include them herein for the benefit of individuals working with the formulae developed for parabolic arches.

(a) To determine if formulae worked out by Iselin and LaLande based on the basic formula of a parabola, are applicable to all parabolas regardless of rise to span ratio:

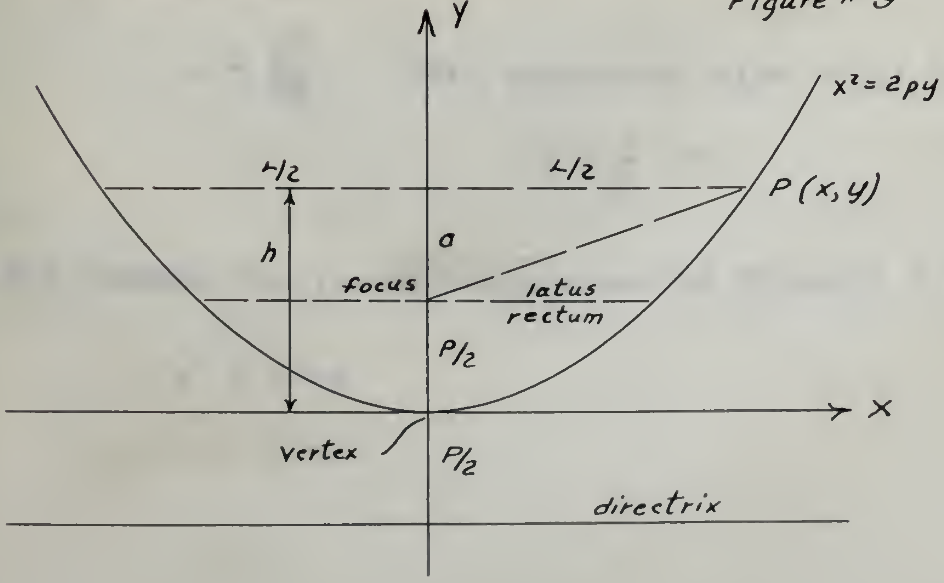


Figure #9

A parabola is the locus of a point such that its distance from a fixed point is equal to its distance from a fixed line. Referring to Figure # 9 and the terminology used-

Let $h = \frac{p}{2} + a$

Since the point p (x,y) must be equidistant from the directrix and focus,

$$\sqrt{a^2 + \frac{L^2}{4}} = a + \frac{p}{2} + \frac{p}{2} = a + p$$

$$a^2 + \frac{L^2}{4} = a^2 + 2\ ap + p^2$$

$$\frac{L^2}{4} = 2\ ap + p^2 = p\ (2a + p)$$

but, $h = a + \frac{p}{2} = \frac{2a + p}{2}$

$$\therefore \frac{L^2}{4} = p\ 2h$$

$p = \frac{L^2}{8h}$ This expression also holds true if $h = \frac{p}{2} - a$

The basic formula for the parabola shown in Figure # 9 is,

$$x^2 = 2\ py$$

$$x^2 = 2\,\frac{L^2}{8h}\;y = \frac{L^2}{4h}\;y$$

$$y = \frac{x^2}{k} \quad \text{where } k = \frac{L^2}{4h}$$

This is the basic expression used by Iselin and LaLande for the derivation of parabolic arch formulae. By referring to Figure # 9, it can be readily seen that any ratio of rise to span may be obtained by varying "h". Since in the above expression "h" can vary from zero to infinity, by changing the value of "a", the expression holds true for symmetrical parabolic arches of any rise to span ratio.

(b) To determine if one, and only one, parabolic arch is defined by any given rise, "h", and span, "L". Knowing the rise and span of the arch, three points on the parabola are determined as follows:

$$\left(-\frac{L}{2},\, h\right), \quad (0,\, 0), \quad \left(\frac{L}{2},\, h\right)$$

Referring to Figure # 9 and the accompanying discussion, the above three (3) points will define at least one symmetrical parabola whose equation is $x^2 = \frac{L^2}{4h}\,y$. This equation was developed using the geometrical relationship provided by the basic definition of a parabola, namely that $2p = \frac{L^2}{4h}$. This

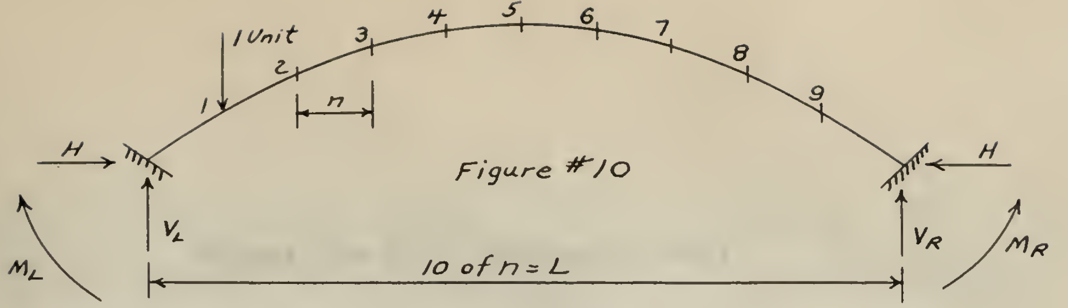
fixes the position of the focus, vertex, and directrix. Any curve drawn equidistant from the focus and directrix, and passing through the above three (3) points, would therefore coincide with the given symmetrical parabola. Accordingly, only one symmetrical parabolic arch is defined by any given rise and spar.

CONSTANTS FOR UNIT CIRCULAR AND PARABOLIC ARCHES

Applying the formulae developed herein, as illustrated on Pages 24 through 28, influence line ordinate values of H, M_R, and M_L are computed for the unit circular arch, ie, an arch whose span is one foot. These influence line values for the circular arch of both constant and variable moment of inertia are tabulated on Page 53. Similar values as derived for the parabolic arch by Iselin and LaLande are tabulated on Page 54.

In comparing the tabulated results for parabolic and circular arches of constant and variable moment of inertia, it is interesting to note that values for arches with a rise to span ratio of approximately 0.12 or less are within 5% of mean values for the four arches. Accordingly, for arches with a rise to span ratio of less than 0.12, the solution of any type arch could be determined by the simple formulae derived by Iselin and LaLande for parabolic arches with a varying moment of inertia.

As stated in the introduction, the purpose of this thesis is to develop a series of simple curves which will lead to the direct solution of both parabolic and circular arches. Since a simple series of influence line curves cannot be developed, because of the variation in values above rise to span ratios of 0.12, a different method will be used in presenting results. Using the tabulated values for



Values for Unit Circular Arch

Rise to Span Ratio	Load Point	H		M_R		M_L	
		Const. I	Vary. I	Const. I	Vary. I	Const. I	Vary. I
	1	0.7708	0.757	0.0115	0.0112	-0.0604	-0.0608
	2	2.4309	2.386	0.0328	0.0317	-0.0630	-0.0643
0.04	3	4.1667	4.116	0.0481	0.0469	-0.0356	-0.0371
	4	5.4367	5.376	0.0491	0.0475	0.0012	-0.0005
	5	5.8931	5.833	0.0323	0.0308	0.0323	0.0308
	1	0.3875	0.3827	0.0116	0.0115	-0.0600	-0.0605
	2	1.2134	1.2064	0.0326	0.0327	-0.0625	-0.0633
0.08	3	2.0777	2.0738	0.0478	0.0482	-0.0351	-0.0358
	4	2.7038	2.7054	0.0486	0.0490	0.0013	0.0010
	5	2.9304	2.9344	0.0221	0.0323	0.0321	0.0323
	1	0.1707	0.1607	0.0131	0.0131	-0.0562	-0.0589
	2	0.5100	0.4987	0.0351	0.0366	-0.0549	-0.0594
0.20	3	0.8466	0.8486	0.0501	0.0537	-0.0273	-0.0303
	4	1.0828	1.1005	0.0508	0.0553	0.0070	0.0073
	5	1.1669	1.1914	0.0355	0.0388	0.0355	0.0388
	1	0.1311	0.1155	0.0156	0.0157	-0.0496	-0.0563
	2	0.3628	0.3503	0.0386	0.0429	-0.0435	-0.0521
0.30	3	0.5778	0.5880	0.0532	0.0627	-0.0163	-0.0213
	4	0.7233	0.7572	0.0541	0.0658	0.0151	0.0178
	5	0.7743	0.8178	0.0405	0.0498	0.0405	0.0498
	1	0.1179	0.0971	0.0193	0.0199	-0.0290	-0.0521
	2	0.2931	0.2859	0.0430	0.0532	-0.0282	-0.0428
0.40	3	0.4443	0.4731	0.0572	0.0781	-0.0022	-0.0059
	4	0.5432	0.6049	0;0585	0.0844	0.0255	0.0364
	5	0.5774	0.6519	0.0472	0.0694	0.0472	0.0694

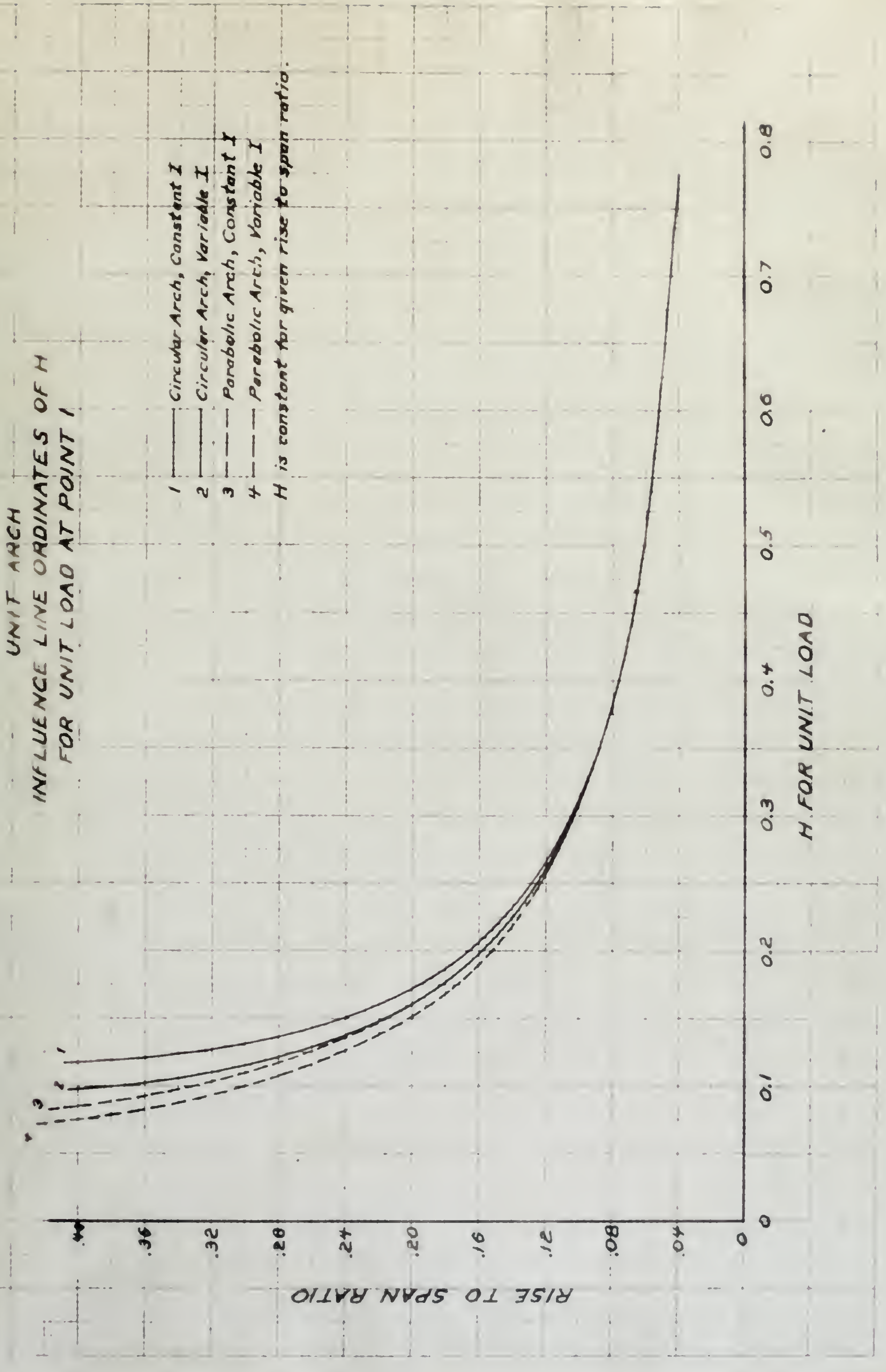
Values for Unit Parabolic Arch

Rise to Span Ratio	Load Point	H		M_E		M_L	
		Const. I	Vary. I	Const. I	Vary. I	Const. I	Vary. I
0.04	1	0.77	0.76	.0116	.0113	-.0604	-.0605
	2	2.41	2.40	.0325	.0320	-.0633	-.0640
	3	4.14	4.14	.0478	.0473	-.0359	-.0367
	4	5.40	5.40	.0487	.0480	.0008	.0000
	5	5.86	5.86	.0320	.0317	.0320	.0317
0.08	1	0.39	0.38	.0118	.0113	-.0600	-.0605
	2	1.21	1.20	.0326	.0320	-.0627	-.0640
	3	2.07	2.07	.0477	.0473	-.0354	-.0367
	4	2.70	2.70	.0484	.0480	.0010	.0000
	5	2.93	2.93	.0319	.0317	.0319	.0317
0.20	1	0.16	0.15	.0117	.0113	-.0589	-.0605
	2	0.49	0.48	.0318	.0320	-.0603	-.0640
	3	0.83	0.83	.0460	.0473	-.0334	-.0367
	4	1.07	1.08	.0463	.0480	.0013	.0000
	5	1.16	1.17	.0303	.0317	.0303	.0317
0.30	1	0.11	0.10	.0116	.0113	-.0579	-.0605
	2	0.33	0.32	.0308	.0320	-.0585	-.0640
	3	0.55	0.55	.0440	.0473	-.0321	-.0367
	4	0.71	0.72	.0434	.0480	.0012	.0000
	5	0.76	0.78	.0282	.0317	.0282	.0317
0.40	1	0.085	0.076	.0116	.0113	-.0569	-.0605
	2	0.253	0.240	.0309	.0320	-.0562	-.0640
	3	0.416	0.414	.0432	.0473	-.0305	-.0367
	4	0.528	0.540	.0423	.0480	.0016	.0000
	5	0.566	0.586	.0276	.0317	.0276	.0317

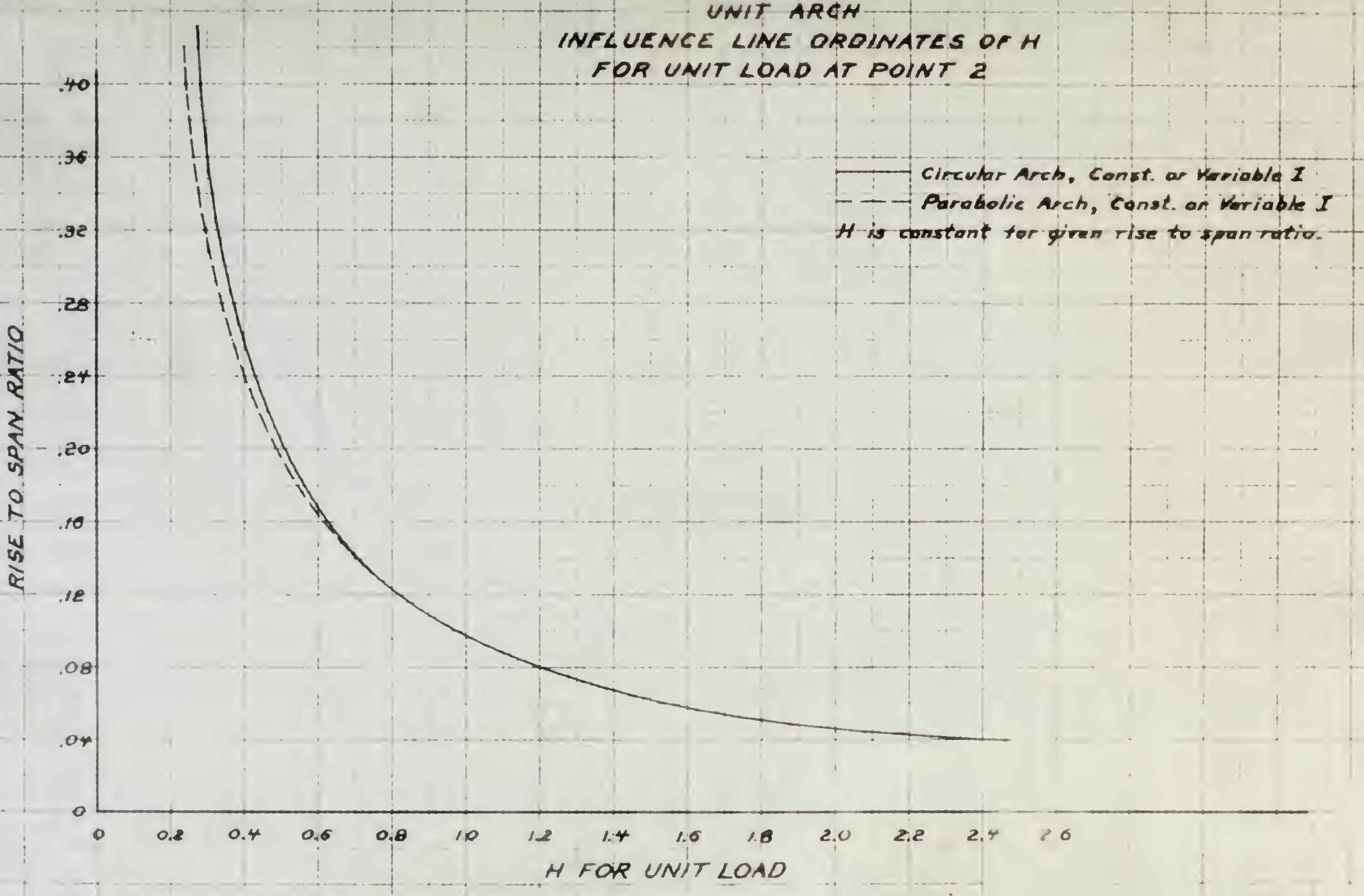
circular and parabolic arches, curves of influence line ordinates for each load point of the circular and parabolic arches will be plotted. These curves are developed by plotting the rise to span ratio against the redundant reaction for each load point; a total of 56 curves, since there are 5 loading points necessary to determine H and 9 loading points necessary to determine M_R and M_L. A curve must be plotted for each of the four types of arches, parabolic and circular with constant and variable moment of inertia. The four curves for each arch, of a redundant value for any given load point, are plotted on the same graph. The proximity of these curves indicate the closeness of results for the various type arches.

The use of these curves is quite simple. Knowing the type of arch and its rise to span ratio, the influence line ordinates for H, M_R, and M_L for each load point can be found directly by use of the 14 graphs. The values of H for any given rise to span ratio remain constant, so they may be plotted as the influence line of H for the given arch. The values of M_R are proportional the span length in feet, and therefore must be multiplied by the span length of the arch to give the correct influence line for M_R. Values of M_L for point 1 are the same as M_R for point 9, etc., therefore once M_R is known, M_L is also known.

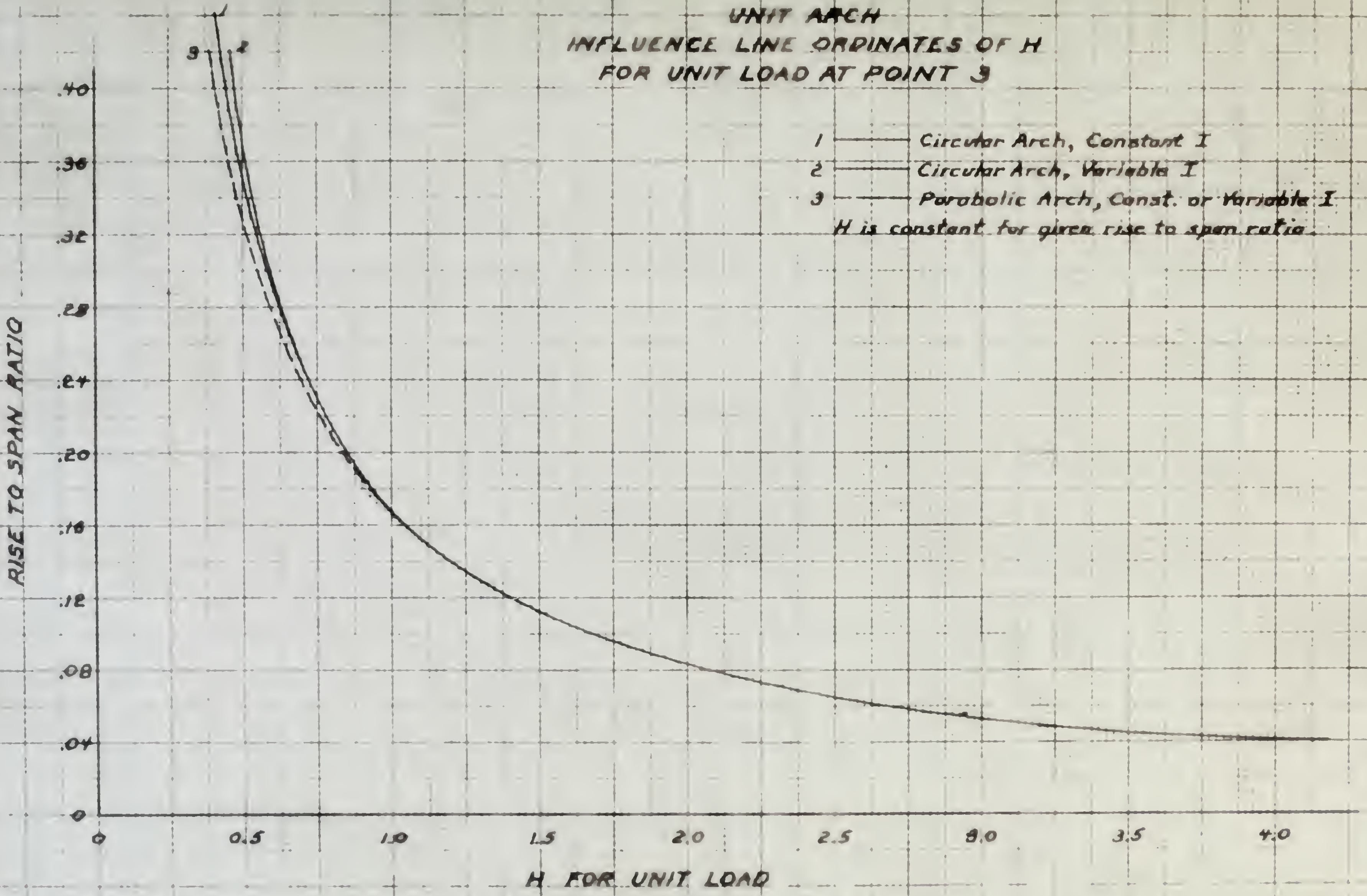
Having obtained 14 simple graphs which will afford a direct solution for the influence lines of symmetrical fixed ended parabolic and circular arches of constant or variable moment of inertia, the work of this thesis is concluded.



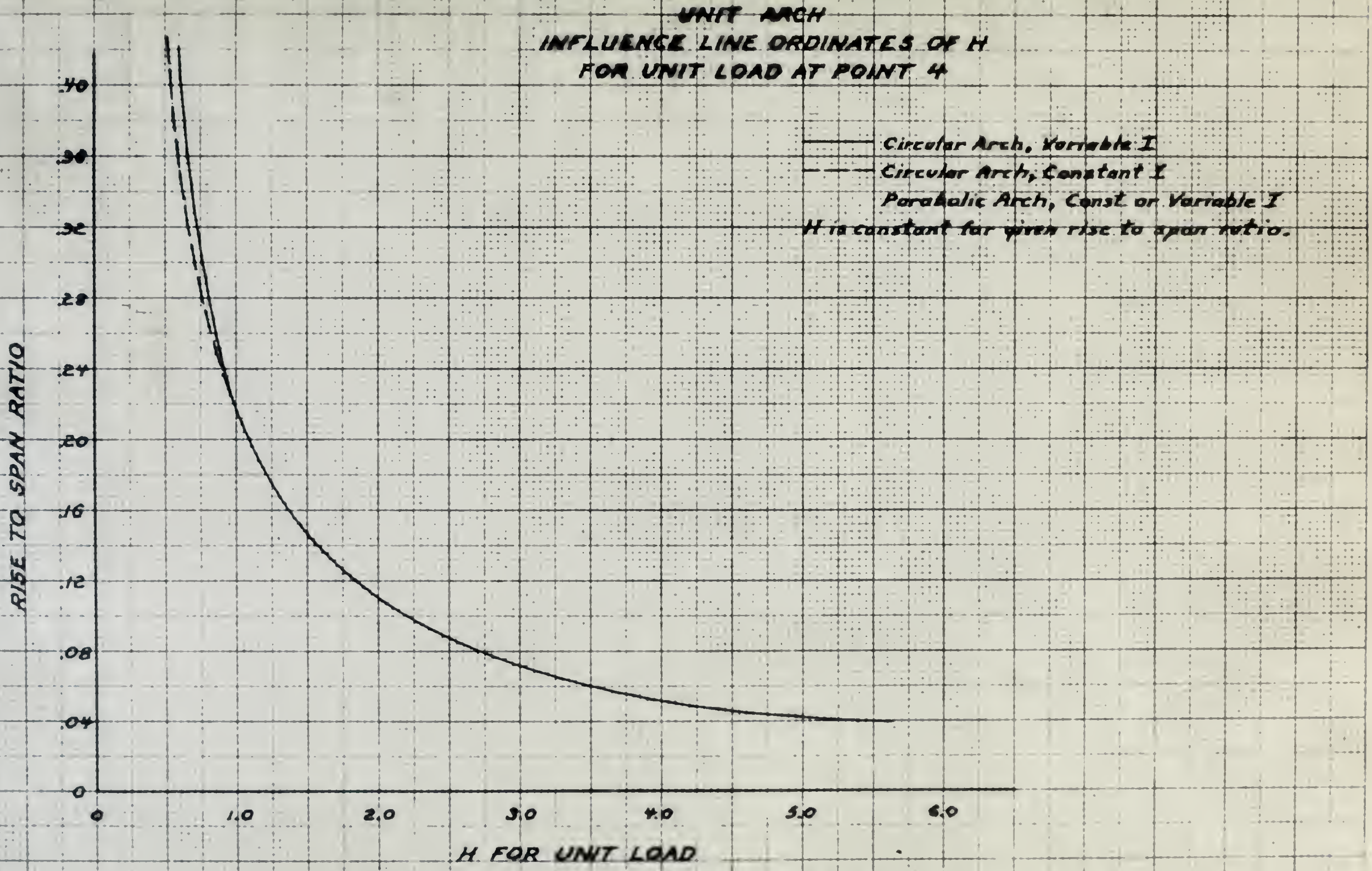
UNIT ARCH
INFLUENCE LINE ORDINATES OF H
FOR UNIT LOAD AT POINT 1



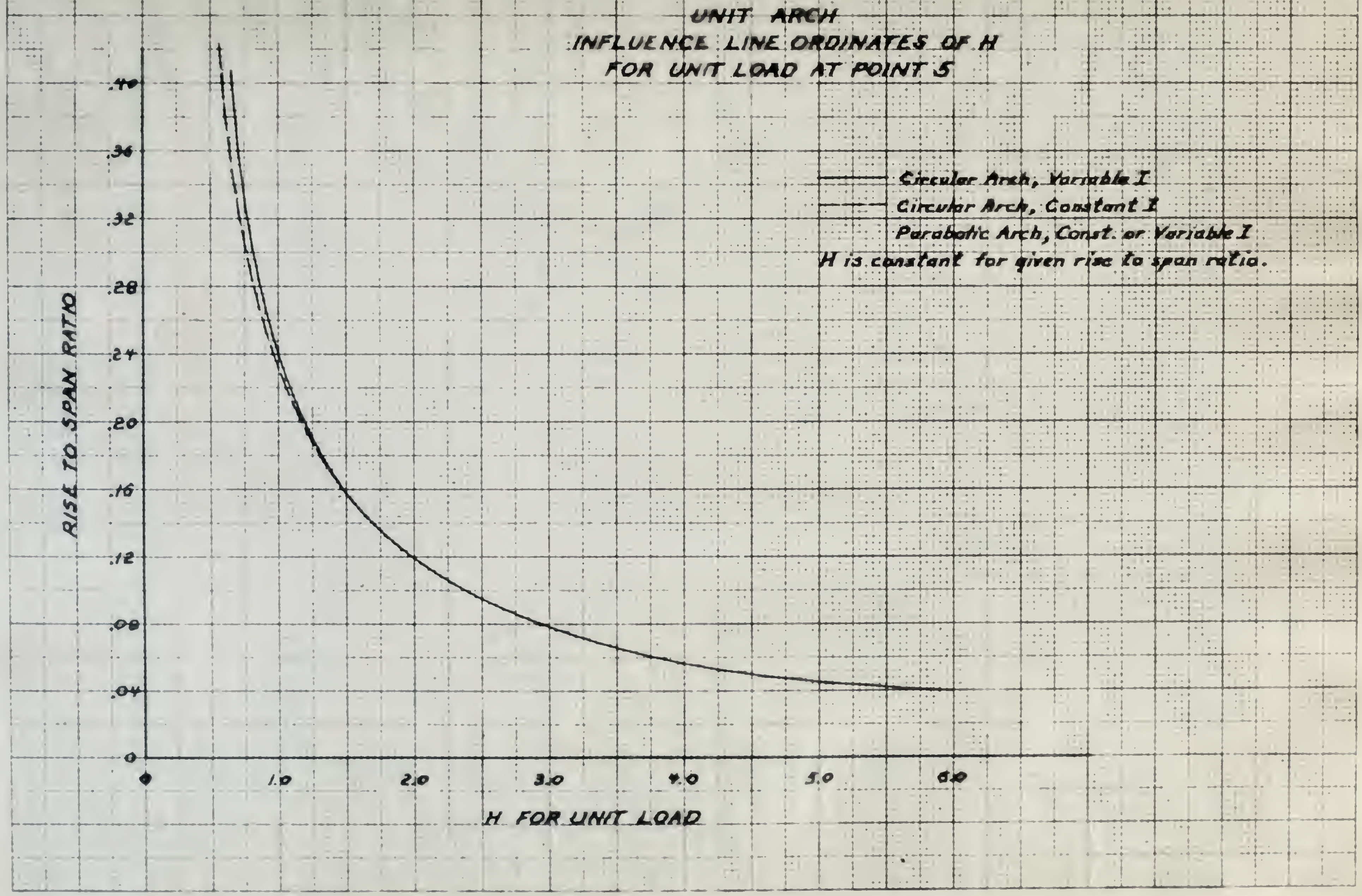
UNIT ARCH
INFLUENCE LINE ORDINATES OF H
FOR UNIT LOAD AT POINT 2



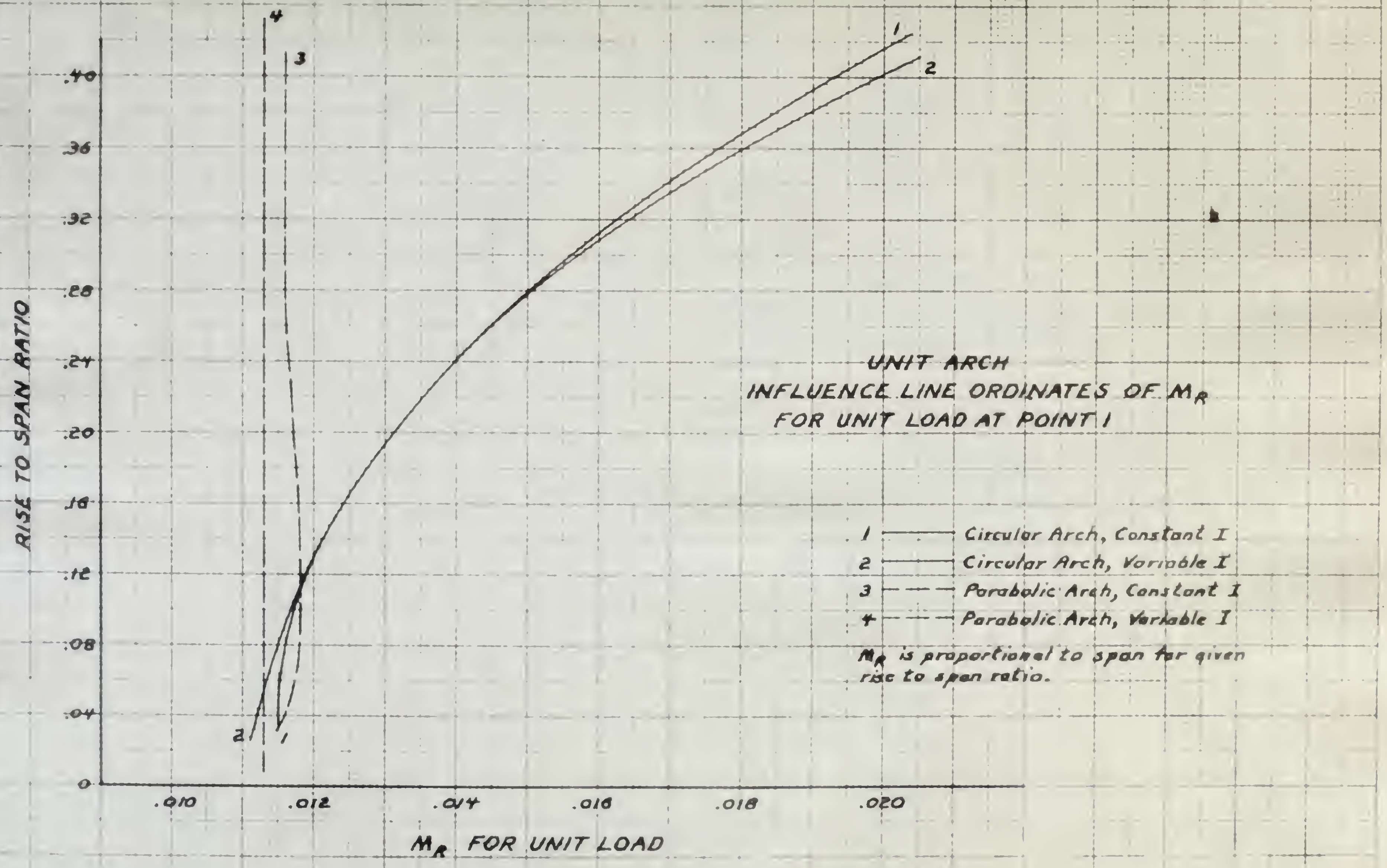
UNIT ARCH
INFLUENCE LINE ORDINATES OF H
FOR UNIT LOAD AT POINT 3



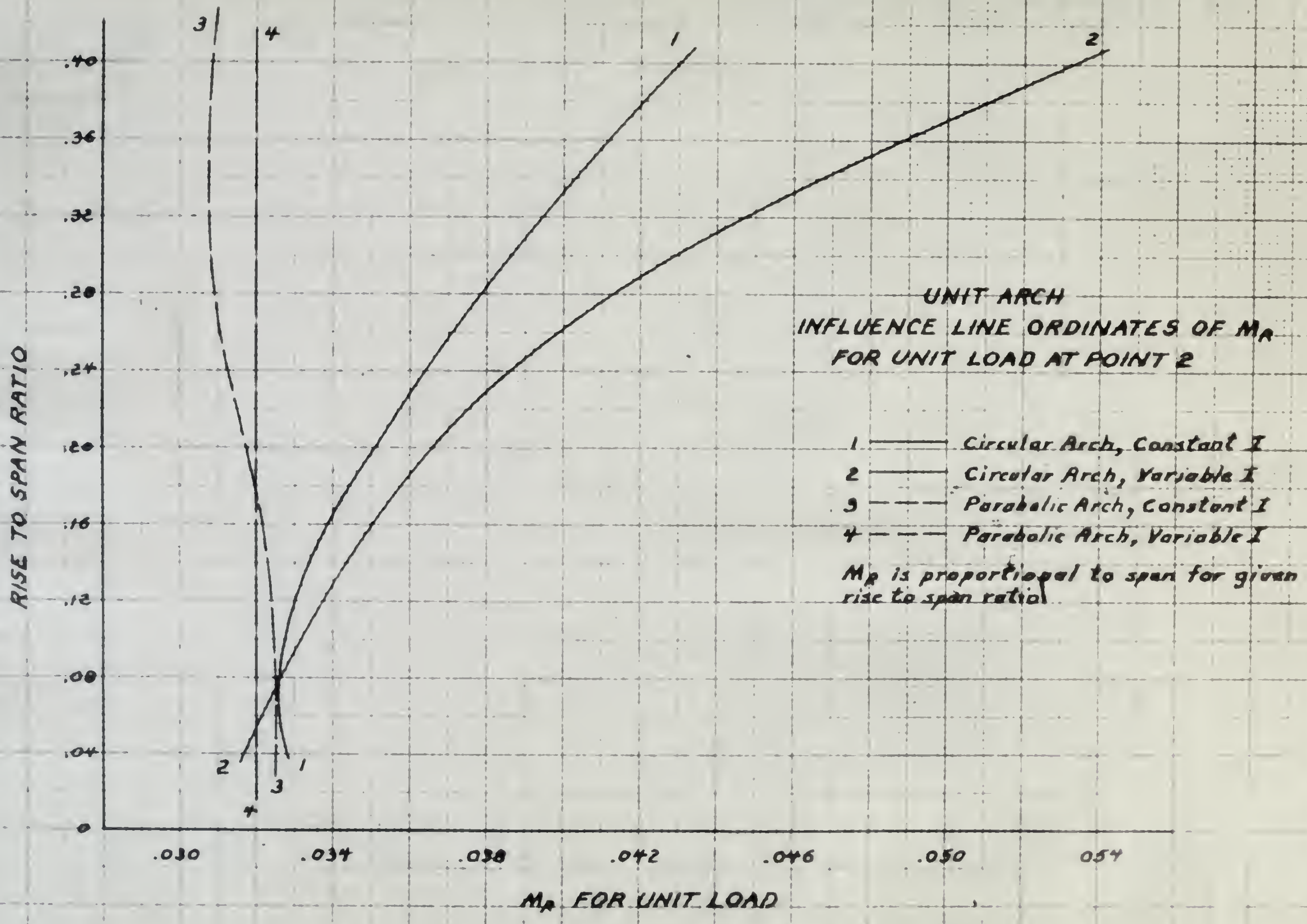
UNIT ARCH
INFLUENCE LINE ORDINATES OF H
FOR UNIT LOAD AT POINT 4



UNIT ARCH
INFLUENCE LINE ORDINATES OF H
FOR UNIT LOAD AT POINT 5



UNIT ARCH
INFLUENCE LINE ORDINATES OF M_R
FOR UNIT LOAD AT POINT 1

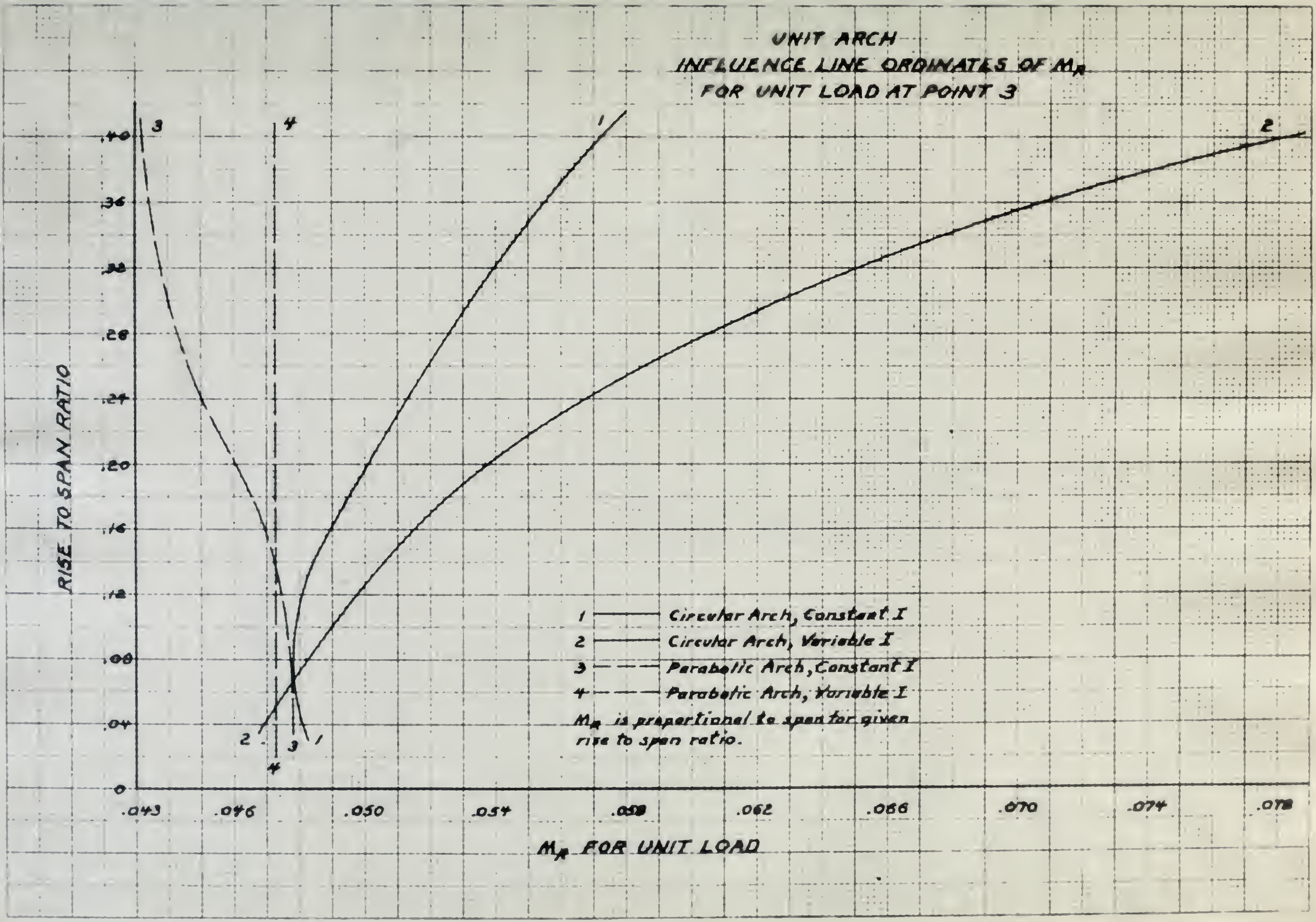


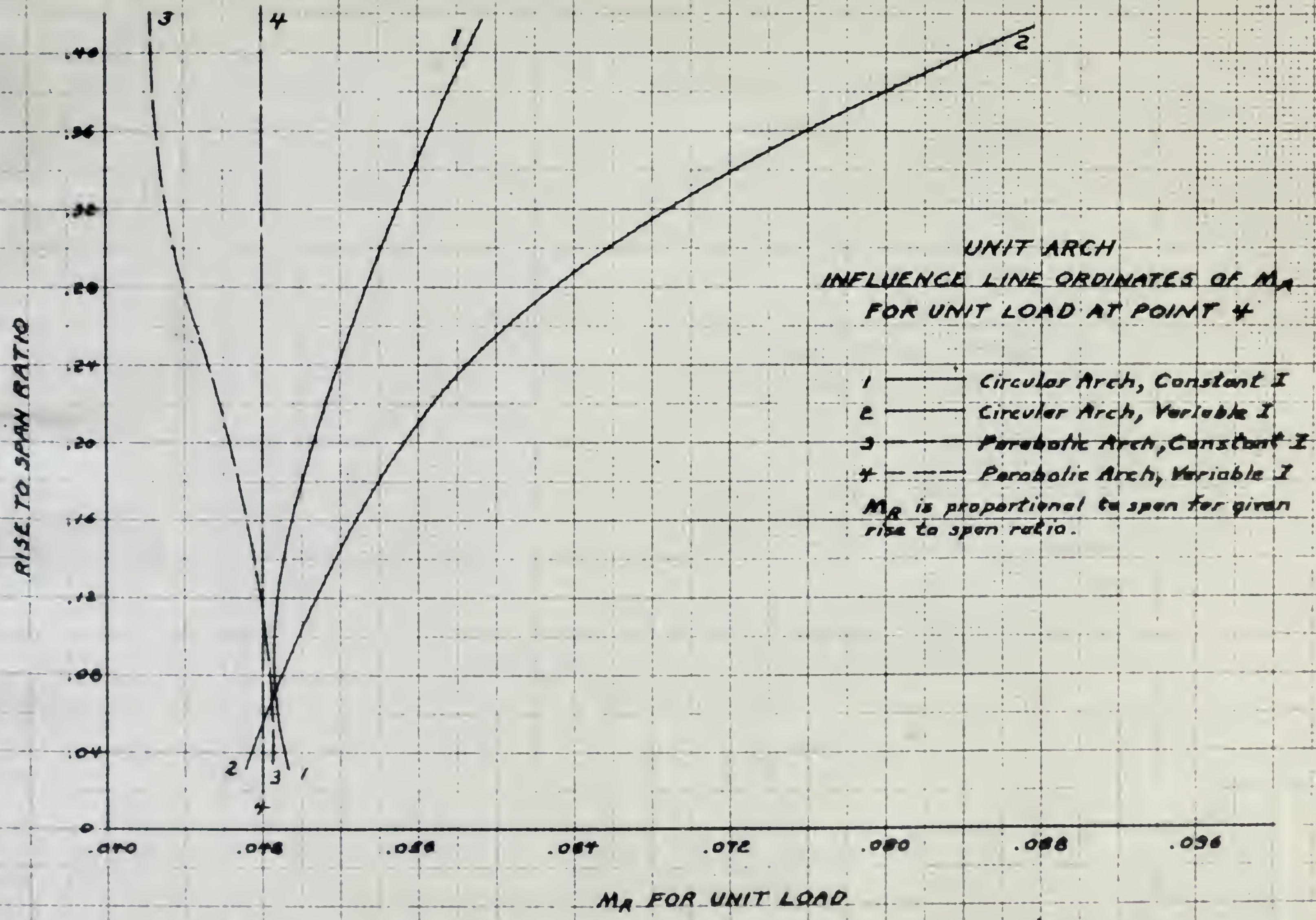
UNIT ARCH
INFLUENCE LINE ORDINATES OF M_R
FOR UNIT LOAD AT POINT 2

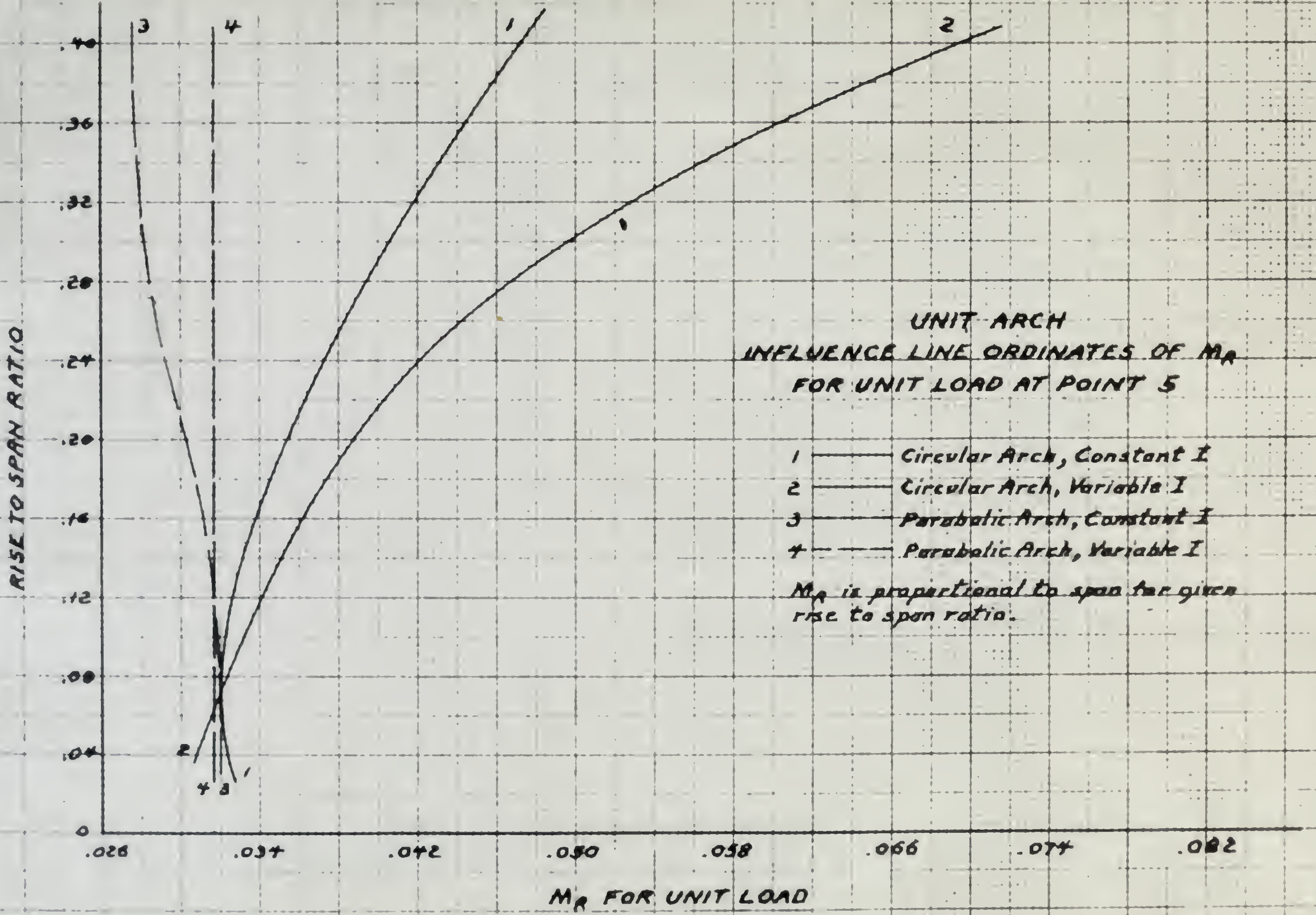
1 ———— Circular Arch, Constant I
2 ———— Circular Arch, Variable I
3 – – – – Parabolic Arch, Constant I
4 — — — Parabolic Arch, Variable I

M_R is proportional to span for given rise to span ratio

UNIT ARCH
INFLUENCE LINE ORDINATES OF M_R FOR UNIT LOAD AT POINT 3

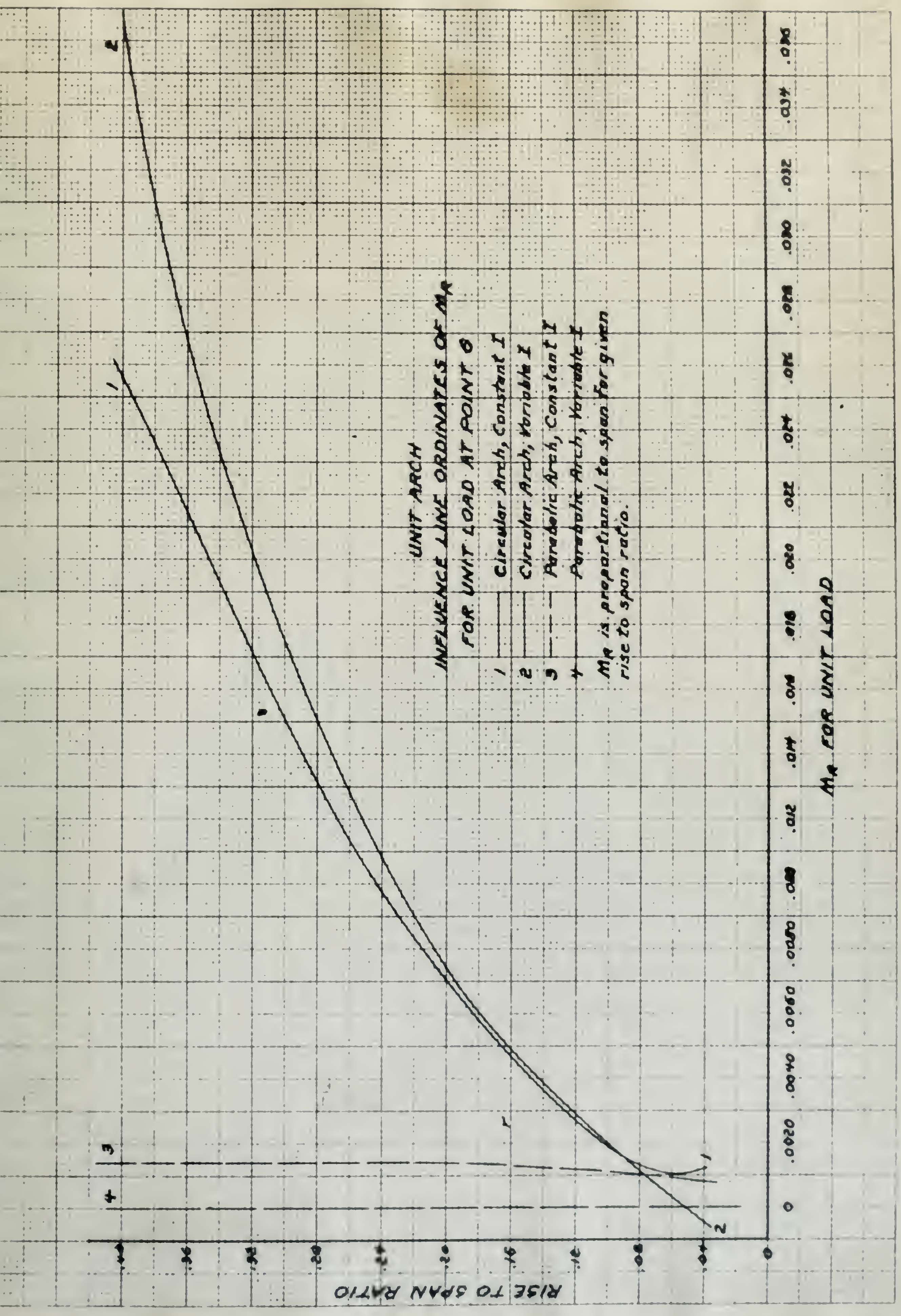


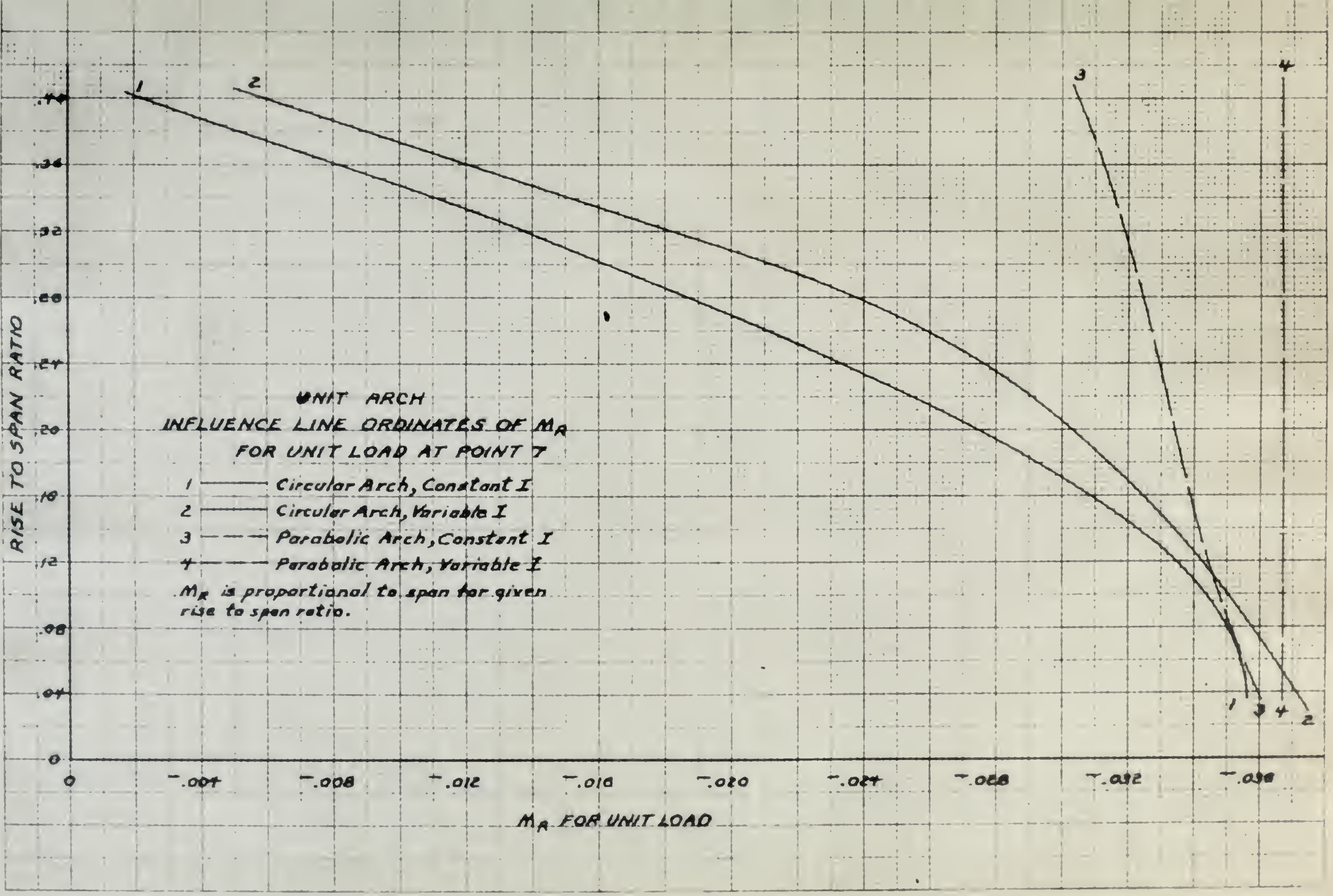

UNIT ARCH
INFLUENCE LINE ORDINATES OF M_A
FOR UNIT LOAD AT POINT 4
1 Circular Arch, Constant I
2 Circular Arch, Variable I
3 Parabolic Arch, Constant I
4 Parabolic Arch, Variable I
M_R is proportional to span for given rise to span ratio.
RISE TO SPAN RATIO
0
.04
.08
.12
.16
.20
.24
.28
.32
.36
.40
.040
.048
.056
.064
.072
.080
.088
.096
M_R FOR UNIT LOAD
1
2
3
4

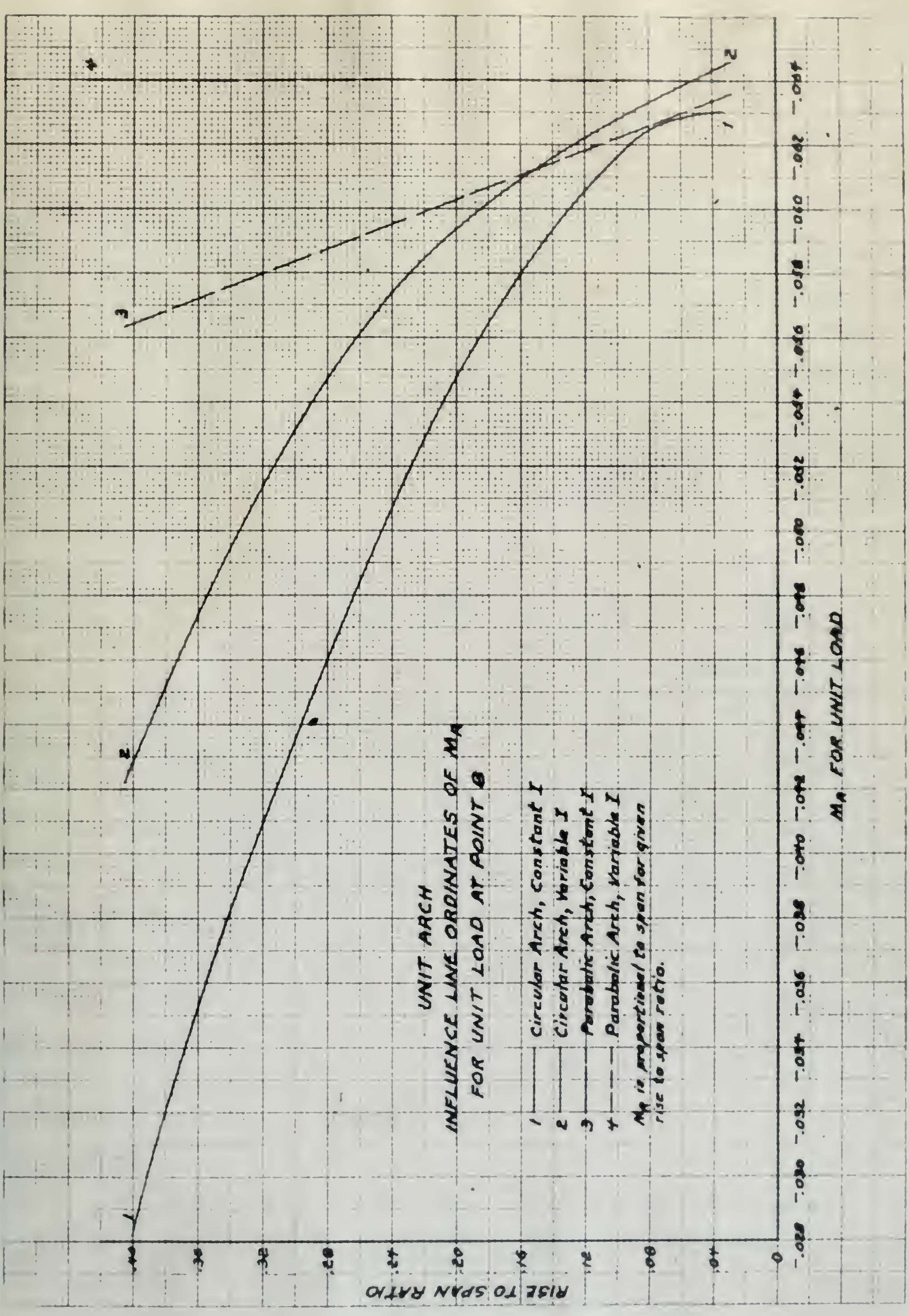


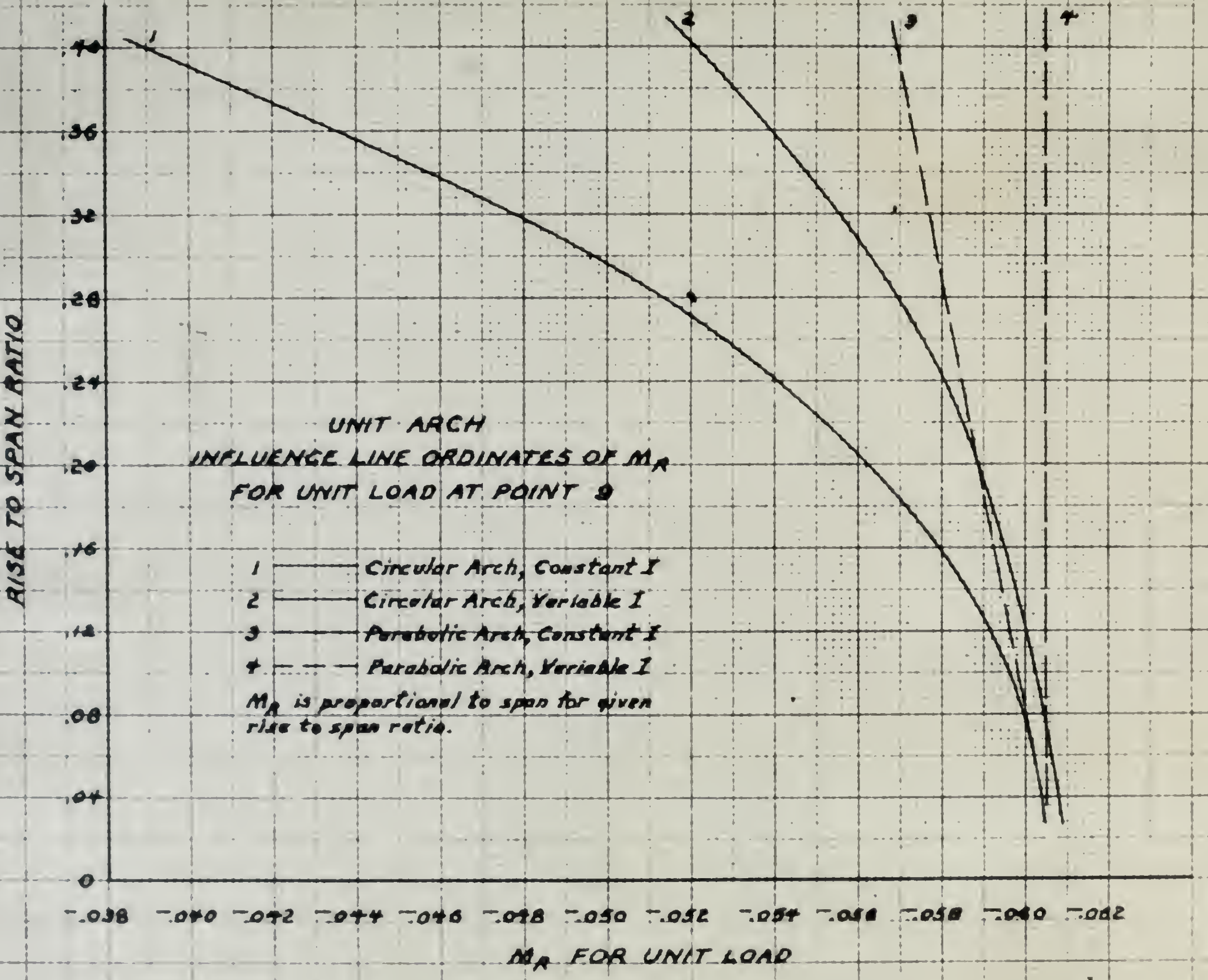
UNIT ARCH
INFLUENCE LINE ORDINATES OF M_A
FOR UNIT LOAD AT POINT 5

M_A is proportional to span for given rise to span ratio.


UNIT ARCH
INFLUENCE LINE ORDINATES OF M_A
FOR UNIT LOAD AT POINT 0
1 — Circular Arch, Constant I
2 — Circular Arch, Variable I
3 — Parabolic Arch, Constant I
4 — Parabolic Arch, Variable I
M_A is proportional to span for given rise to span ratio.
M_A FOR UNIT LOAD
RISE TO SPAN RATIO
0
.04
.08
.12
.16
.20
.24
.28
.32
.36
.40
0
.0020
.0040
.0060
.0080
.010
.012
.014
.016
.018
.020
.022
.024
.026
.028
.030
.032
.034
.036
1
2
3
4


UNIT ARCH
INFLUENCE LINE ORDINATES OF M_A
FOR UNIT LOAD AT POINT 7
1 —— Circular Arch, Constant I
2 —— Circular Arch, Variable I
3 – – – Parabolic Arch, Constant I
4 –·–·– Parabolic Arch, Variable I
M_A is proportional to span for given rise to span ratio.
RISE TO SPAN RATIO
0
.04
.08
.12
.16
.20
.24
.28
.32
.36
.40
0
−.004
−.008
−.012
−.016
−.020
−.024
−.028
−.032
−.036
M_A FOR UNIT LOAD
1
2
3
4


UNIT ARCH
INFLUENCE LINE ORDINATES OF M_A
FOR UNIT LOAD AT POINT B
1 —— Circular Arch, Constant I
2 —— Circular Arch, Variable I
3 —— Parabolic Arch, Constant I
4 —— Parabolic Arch, Variable I
M_A is proportional to span for given rise to span ratio.
RISE TO SPAN RATIO
0
.04
.08
.12
.16
.20
.24
.28
.32
.36
.40
-.028
-.030
-.032
-.034
-.036
-.038
-.040
-.042
-.044
-.046
-.048
-.050
-.052
-.054
-.056
-.058
-.060
-.062
-.064
M_A FOR UNIT LOAD
1
2
3
4
1
2


UNIT ARCH
INFLUENCE LINE ORDINATES OF M_A
FOR UNIT LOAD AT POINT 9
1 Circular Arch, Constant I
2 Circular Arch, Variable I
3 Parabolic Arch, Constant I
4 Parabolic Arch, Variable I
M_A is proportional to span for given rise to span ratio.
1
2
3
4
RISE TO SPAN RATIO
.40
.36
.32
.28
.24
.20
.16
.12
.08
.04
0
-.038
-.040
-.042
-.044
-.046
-.048
-.050
-.052
-.054
-.056
-.058
-.060
-.062
M_A FOR UNIT LOAD

CONCLUSIONS

The series of 14 graphs developed in this thesis, enable the designer to determine immediately exact values of ordinates of influence lines for H, M_L, and M_R of symmetrical fixed end arches. The curves are applicable to the solution of both circular and parabolic arches with a constant cross sectional moment of inertia or a moment of inertia which varies as the secant function times the moment of inertia at the crown. The range of rise to span ratios of from 0.04 to 0.40 should include practically all conceivable arch sections.

The use of the graphs contained herein, eliminate the necessity of solution of arches by the standard methods which are lengthy and tedious in their application and give only approximate results.

It is suggested that further work be directed towards the solution of the above types of arches for horizontal loads, thus affording a complete solution for all usual types of loading.

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